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F19_LN_12_5

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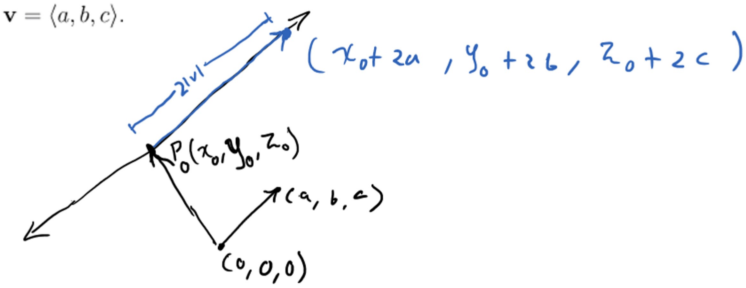
F19_LN_12_5

12.5: Equations of lines and planes

Lines

Lines determined by a point and a vector

Consider line L that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b, c \rangle$.



Parametric equations of the line:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

EXAMPLE 1. *Find parametric equations of the line*

(a) *passing through the point* $(3, -4, 1)$ *and parallel to* $\mathbf{v} = \langle 7, 0, -1 \rangle$

$x = 3 + 7t$
$y = -4 + 0 \cdot t = -4$
$z = 1 - t$

(b) *passing through the origin and parallel to* $\mathbf{v} = \langle 5, 5, 5 \rangle$

$x = 0 + 5t = 5t$
$y = 5t$
$z = 5t$

EXAMPLE 2. *Consider the line L that passes through the points $A(1, 1, 1)$ and $B(2, 3, -2)$. Find points at that L intersects the yz-plane.*

$\overrightarrow{AB} = (2-1, 3-1, -2-1) = \langle 1, 2, -3 \rangle$

A point (x,y,z) on the line L satisfies

$x = 1 + t$
$y = 1 + 2t$
$z = 1 - 3t$

(x,y,z) is in the yz-plane if $x=0$, so $1+t = x = 0 \iff t = -1$.

So L intersect the plane at $(x,y,z) \overset{(t=-1)}{=} (0,-1,4)$.

Symmetric equations of the line: If $abc \neq 0$ then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$$

If, for example, $a = 0$ then the symmetric equations have the form:

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

EXAMPLE 3. *Find symmetic equations of lines from Example 1.*

a) Passing through $(x_0, y_0, z_0) = (3, -4, 1)$, parallel to $\langle 7, 0, -1\rangle = \langle a, b, c\rangle$

$$\boxed{\frac{x-3}{7} = \frac{z-1}{-1}, \quad y = y_0}$$

Notice, $b = 0$

b) The line passing through $(0,0,0)$ parallel $\langle 5,5,5\rangle$ is given by the symmetric $\frac{x}{5} = \frac{y}{5} = \frac{z}{5} \iff \boxed{x = y = z}$

Vector equation of the line:

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle$$

where $P_0(x_0, y_0, z_0)$ is a given point on the line and $\mathbf{v} = \langle a, b, c\rangle$ is some vector which is parallel to the line, t is a parameter, $\underbrace{-\infty < t < \infty}_{t \in \mathbb{R}}$.

EXAMPLE 4. *Find vector equation of the line that passes through the points $P(1, 1, -4)$ and $Q(0, 3, -4)$.*

The displacement vector $\overrightarrow{PQ} = \langle 0-1, 3-1, -4-(-4)\rangle = \langle -1, 2, 0\rangle$ is parallel to the line. The vector equation is

$$\boxed{r(t) = \langle 1, 1, -4\rangle + t\langle -1, 2, 0\rangle}$$

EXAMPLE 5. *Determine whether the lines*

$$L_1: \quad x-1 = \frac{y+2}{3} = \frac{z-4}{-1}$$

Given by Symmetric eqns.

and

$$L_2: \quad x = 2t, \quad y = 3+t, \quad z = -3+4t$$

Given by Parametric eqns.

are parallel, skew, or intersecting.

This step is not necessary

To find vectors parallel to each line, let's first find 2 points on each line

Points on L_1: Solving $0 = x-1 = \frac{y+2}{3} = \frac{z-4}{-1}$ and $1 = x-1 = \frac{y+2}{3} = \frac{z-4}{-1}$ for (x,y,z), we get two points on L_1

$$P_1 = (1,-2,4) \quad \text{and} \quad Q_1 = (2,1,3)$$

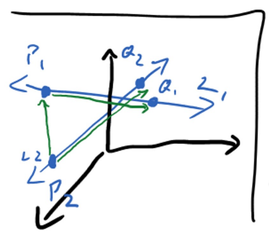
Points on L_2: Choosing $t=0$ and $t=1$, we get 2 points on L_2

$$P_2 = (0,3,-3) \quad \text{and} \quad Q_2 = (2,4,1).$$

$\overrightarrow{P_1Q_1} = \langle 1, 3, -1 \rangle$ and $\overrightarrow{P_2Q_2} = \langle 2, 1, 4 \rangle$ are parallel to L_1 and L_2 respectively.

$\overrightarrow{P_1Q}$ is not a scalar multiple of $\overrightarrow{P_2Q_2}$, so $\boxed{L_1 \text{ and } L_2 \text{ are not parallel}}$

$\overrightarrow{P_2P_1} = \langle 1, -5, 7 \rangle.$



$$\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\hat{i} - 6\hat{j} - 5\hat{k}.$$

$$\overrightarrow{P_2P_1} \cdot (\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}) = 13 + 30 - 35 \neq 0$$

Therefor $\overrightarrow{P_2P_1}$, $\overrightarrow{P_1Q_1}$, and $\overrightarrow{P_2Q_2}$ are not coplanar, so $\boxed{L_1 \text{ and } L_2 \text{ are skew}}$

Summarizing table

Vector equation	Parametric equations	Symmetric equations
$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$, $-\infty < t < \infty$	$x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, $-\infty < t < \infty$	If $abc \neq 0$ then $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ If, for example, $a = 0$ then the symmetric equations have the form: $x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$

Line segments

How to find parametric equation of a line segment:

1. *Find parametric equation for the entire line;*
2. *restrict the parameter appropriately so that only the desired segment is generated.*

EXAMPLE 6. *Find parametric equations describing the line segment joining the points* $M(1,2,3)$ *and* $N(3,2,1)$.

$\overrightarrow{MN} = \langle 2, 0, -2 \rangle$, so par. eqns. are

$$x = 1 + 2t$$
$$y = 2 + 0t = 2$$
$$z = 3 + (-2)t = 3 - 2t.$$

If $t = 0$ then $M = (x, y, z)$. Increasing t moves the point (x, y, z) along the line towards N.

To find t so that $N = (x, y, z)$, we need $3 = x = 1 + 2t$, that is, $t = 1$, so the line segment is defined by

$$\boxed{\left.\begin{array}{l} x = 1 + 2t \\ y = 2 \\ z = 3 - 2t \end{array}\right\} \text{ with } 0 \le t \le 1}$$

Planes

Planes parallel to the coordinate planes:

Planes parallel to the xy-plane are described by the equation $z=k$ for some # k.

" " " " xz-plane " " " " " $y=k$ ".

" " " " yz-plane " " " " " $x=k$ ".

Planes determined by a point and a normal vector

A plane in $\mathbb{R}^3$ is uniquely determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n} = (a, b, c)$ that is orthogonal to the plane. This vector is called a **normal vector**.

Assume that $P(x, y, z)$ is any point in the plane. Let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors for P_0 and P respectively.

Vector equation of the plane: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$.

Scalar equation of plane:

$$\underbrace{a(x - x_0) + b(y - y_0) + c(z - z_0)}_{= ax+by+cz-(ax_0+by_0+cz_0)} = 0.$$

Often this will be written as a **linear equation** in x, y, z,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

EXAMPLE 7. *Determine the equation of the plane through the point* $(1, 2, 1)$ *and orthogonal to vector* $\langle 2, 3, 4\rangle$. *Find the intercepts and sketch the plane.*

Either eqn. describes the plane

$$2(x-1) + 3(y-2) + 4(z-1) = 0.$$

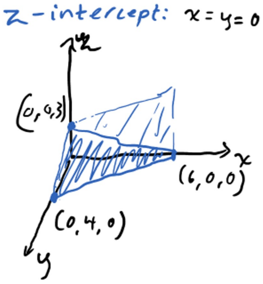
$$2x + 3y + 4z = 12$$

This arrow means "implies"

Find x-intercept by setting $y=z=0$: $y=z=0 \Rightarrow 2x = 12 \Rightarrow \boxed{x = 6}$

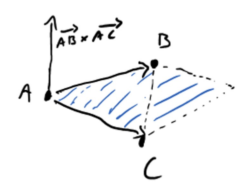
y-intercept: $x=z=0 \Rightarrow 3y = 12 \Rightarrow \boxed{y=4}$

z-intercept: $x=y=0 \Rightarrow 4z = 12 \Rightarrow \boxed{z=3}$



EXAMPLE 8. *Determine the equation of the plane through the points* $A(1, 1, 1)$, $B(0, 1, 0)$ *and* $C(1, 2, 3)$.

$\vec{AB} = \langle -1, 0, -1\rangle$, $\vec{AC} = \langle 0, 1, 2\rangle$



$$n = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \hat{i} + 2\hat{j} - \hat{k}$$

The plane is described by

$$1(x-1) + 2(y-1) + (-1)(z-1) = 0$$

$$\boxed{x + 2y - z = 2}$$

Two planes are **parallel** if their normal vectors are parallel.

Two planes are **orthogonal** if their normal vectors are orthogonal.

If two planes are not parallel, then they intersect in a straight line and the **angle** between the two planes is defined as the *acute* angle between their normal vectors.

EXAMPLE 9. *Given four planes:*

$$\begin{array}{lrcrcrcrcl} P_1: & 2x & + & 3y & + & z & + & 11 & = & 0 \\ P_2: & -4x & - & 6y & - & 2z & + & 77 & = & 0 \\ P_3: & 2x & & & - & 4z & + & 33 & = & 0 \\ P_4: & -2x & + & 3y & + & z & + & 11 & = & 0. \end{array}$$

normal vectors
$n_1 = \langle 2, 3, 1\rangle$
$n_2 = \langle -4, -6, -2\rangle$
$n_3 = \langle 2, 0, -4\rangle$
$n_4 = \langle -2, 3, 1\rangle$

Determine whether the given pairs of the planes are parallel, orthogonal, or neither. Find the angle between the planes.

(a) P_1 *and* P_2 $\quad P_1 \parallel P_2$ because $n_2 = -2\, n_1$.

denotes parallel

$\underbrace{\langle v_1, v_2, v_3\rangle}_{\vec{v}} \cdot \underbrace{\langle w_1, w_2, w_3\rangle}_{\vec{w}} = |\vec{v}||\vec{w}| \cos\theta$

Angle between v and w

(b) P_1 *and* P_3

$n_1 \cdot n_3 = 2\cdot 2 + 3\cdot 0 + 1\cdot(-4) = 0$

Therefore P_1 and P_3 are ortllogonal (ie, $P_1 \perp P_3$)

(c) P_2 *and* P_3 $\quad P_2 \perp P_3$ (by parts a an b above)

(d) P_1 *and* P_4

$n_1 \cdot n_4 = 2\cdot(-2) + 3\cdot 3 + 1\cdot 1 = 6$

$|n_1| = \sqrt{4+9+1} = \sqrt{14}$

$|n_4| = \sqrt{14}$

Letting θ be the angle between n_1 and n_2

$\cos\theta = \dfrac{n_1 \cdot n_4}{|n_1||n_4|} = \dfrac{6}{14} = \dfrac{3}{7}.$

$\cos^{-1}\left(\frac{3}{7}\right) \approx 65^\circ$

Angle between the planes

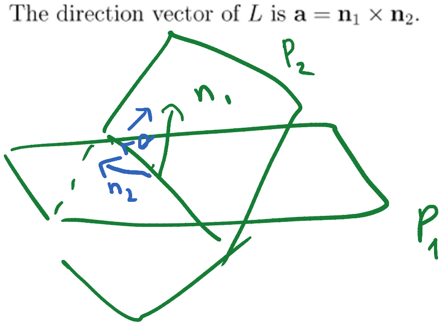
Line as an intersection of two non parallel planes:

$$L:\left\{\begin{array}{lllllllll} a_1x & + & b_1y & + & c_1z & + & d_1 & = & 0 \\ a_2x & + & b_2y & + & c_2z & + & d_2 & = & 0 \end{array}\right.$$

P_1 $\vec{n}_1 = \langle a_1, b_1, c_1 \rangle$

P_2 $\vec{n}_2 = \langle a_2, b_2, c_2 \rangle$

The direction vector of L is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2$.



If $\vec{a}$ is in the direction of L

$\vec{a} \perp \vec{n}_1$

$\vec{a} \perp \vec{n}_2$

We can take $\vec{a} = \vec{n}_1 \times \vec{n}_2$

EXAMPLE 10. *Find an equation of the line given as intersection of two planes:*

$$\left\{\begin{array}{lllllll} x & - & y & + & 3z & = & 0 \\ x & + & y & + & 4z & = & 2 \end{array}\right.$$

$n_1 = \langle 1, -1, 3 \rangle$

$n_2 = \langle 1, 1, 4 \rangle$

$$a = n_1 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = -7\hat{i} - \hat{j} + 2\hat{k} = \langle -7, -1, 2 \rangle$$

We need to find 1 point on the line.

For this we can set $z = 0$: $\left\{\begin{array}{l} x - y = 0 \Rightarrow y = x \\ x + y = 2 \end{array}\right.$ $2x = 2 \Rightarrow x = 1 \Rightarrow y = 1$

$\Rightarrow$ $(1, 1, 0)$ is on our line

The parametric equation of the line is

$$\boxed{\left\{\begin{array}{l} x = 1 - 7t \\ y = 1 - t \\ z = 2t \end{array}\right.}$$