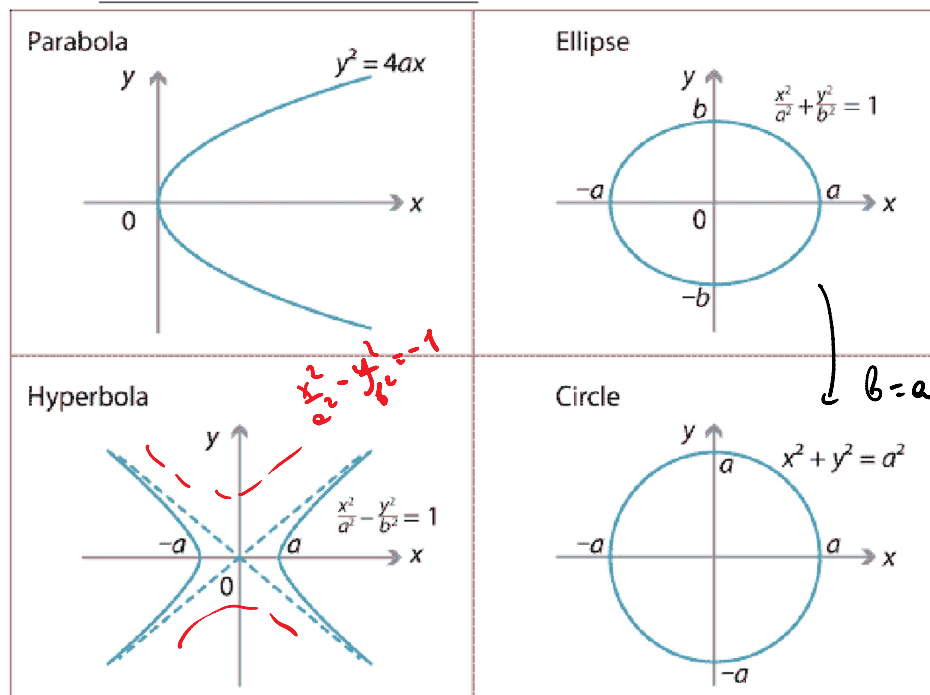


12.6: Quadric surfaces

REVIEW: Parabola, hyperbola and ellipse.



The most general second-degree equation in three variables x, y and z :

$$Ax^2 + By^2 + Cz^2 + axy + bxz + cyz + d_1x + d_2y + d_3z + E = 0, \quad (1)$$

where $A, B, C, a, b, c, d_1, d_2, d_3, E$ are constants. The graph of (1) is a quadric surface.

Note if $A = B = C = a = b = c = 0$ then (1) is a linear equation and its graph is a plane (this is the case of degenerated quadric surface).

By translations and rotations (1) can be brought into one of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

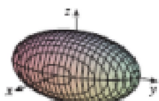
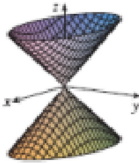

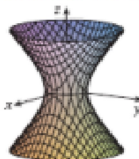
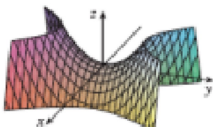
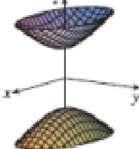
In order to sketch the graph of a surface determine the curves of intersection of the surface with planes parallel to the coordinate planes. The obtained in this way curves are called **traces** or **cross-sections** of the surface.

Quadric surfaces can be classified into 5 categories:

ellipsoids, hyperboloids, cones, paraboloids (quadric cylinders) (shown in the table below.)

was discussed in section 11.1

ellipsoids, hyperboloids, cones, paraboloids (quadric cylinders) (shown in the table below.)

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.</p>	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.</p>
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.</p>	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <p>Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.</p>
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <p>Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.</p>	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.</p>

The elements which characterize each of these categories:

1. Standard equation.
2. Traces (horizontal (by planes $z = k$), yz -traces (by $x = 0$) and xz -traces (by $y = 0$).
3. Intercepts (in some cases).

To find the equation of a trace substitute the equation of the plane into the equation of the surface.

Note, in the examples below the constants a, b , and c are assumed to be positive.

EXAMPLE 1. Use traces to sketch the following quadric surfaces:

(a)

EXAMPLE 1. Use traces to sketch the following quadric surfaces:

(a)

$$x^2 + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Solution

• Find intercepts:

- x -intercepts: if $y = z = 0$ then $x^2 = 1 \Rightarrow x = \pm 1$
- y -intercepts: if $x = z = 0$ then $y^2 = 16 \Rightarrow y = \pm 4$
- z -intercepts: if $x = y = 0$ then $z^2 = 9 \Rightarrow z = \pm 3$

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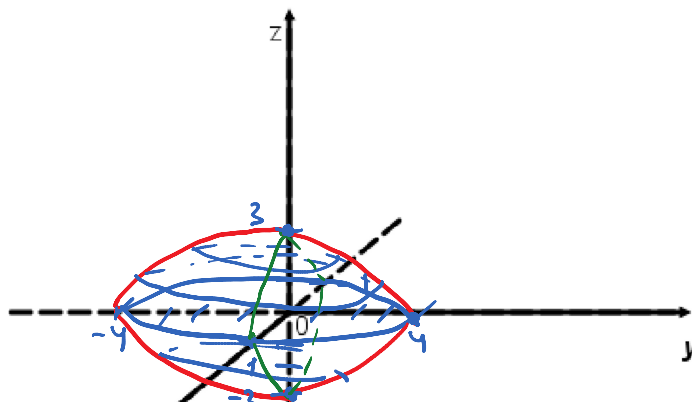
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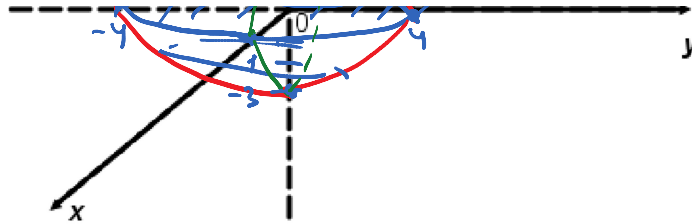
• Obtain traces of:

- the xy -plane: plug in $z = 0$ and get $x^2 + \frac{y^2}{16} = 1 \rightarrow$ ellipse in xy -plane with semiaxes 1 and 4
- the yz -plane: plug in $x = 0$ and get $\frac{y^2}{16} + \frac{z^2}{9} = 1 \rightarrow$ ellipse in yz -plane with semiaxes 4 and 3
- the xz -plane: plug in $y = 0$ and get $x^2 + \frac{z^2}{9} = 1 \rightarrow$ ellipse in xz -plane with semiaxes 1 and 3
- plug in $z = k$ $x^2 + \frac{y^2}{16} + \frac{k^2}{9} = 1 \Rightarrow x^2 + \frac{y^2}{16} = 1 - \frac{k^2}{9} \Rightarrow \frac{x^2}{1 - \frac{k^2}{9}} + \frac{y^2}{16(1 - \frac{k^2}{9})} = 1$
if $-3 < k < 3$ it is ellipse with semiaxes $\sqrt{1 - \frac{k^2}{9}}$ and $4\sqrt{1 - \frac{k^2}{9}}$
- plug in $x = k$
- plug in $y = k$

This is enough to build a surface

work out by yourself





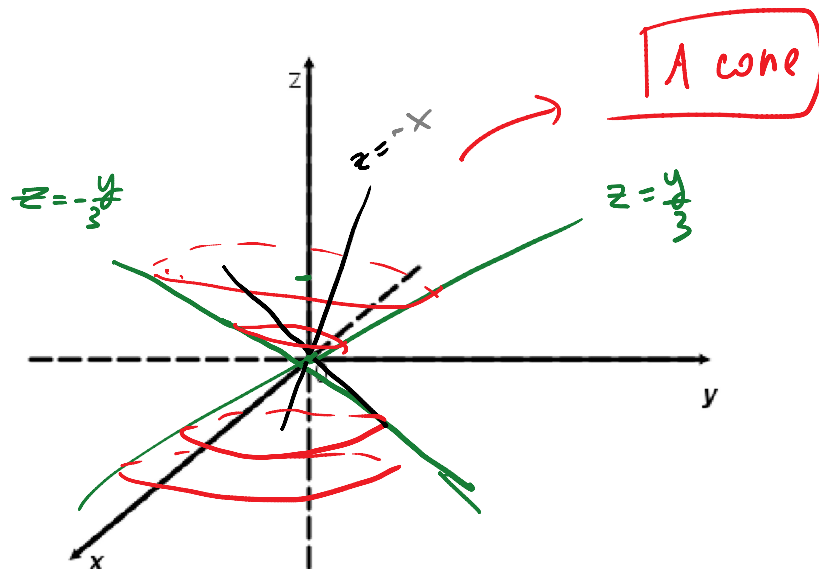
(b)

$$z^2 = x^2 + \frac{y^2}{9}$$

$a^2 = b^2 \Leftrightarrow$
either $a = b$ or
 $a = -b$

Plane	Trace
$z = k$	$x^2 + \frac{y^2}{9} = k^2 \Leftrightarrow \frac{x^2}{k^2} + \frac{y^2}{9k^2} = 1 \Rightarrow$
$x = 0$	$z^2 = \frac{y^2}{9} \Leftrightarrow$ either $z = \frac{y}{3}$ or $z = -\frac{y}{3} \rightarrow$
$y = 0$	$z^2 = x^2 \Leftrightarrow$ either $z = x$ or $z = -x \rightarrow$

ellipse with
semiaxis k and $3k$
2 lines in yz -plane
2 lines in xz -plane



(c)

$$z = \frac{x^2}{4} + \frac{y^2}{9}$$

$k \geq 0$

Plane	Trace
$z = k$	$k = \frac{x^2}{4} + \frac{y^2}{9} \Leftrightarrow \frac{x^2}{4k} + \frac{y^2}{9k} = 1$
$x = 0$	$z = \frac{y^2}{9}$

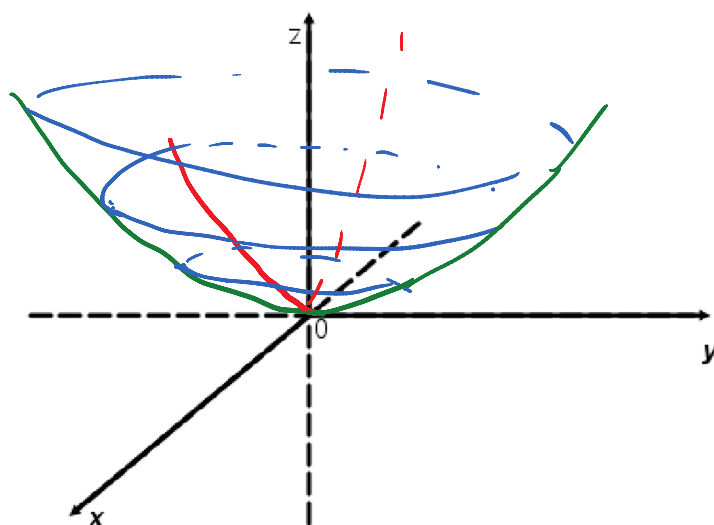
$k > 0 \rightarrow$ ellipse with
semiaxis $2\sqrt{k}$ and $3\sqrt{k}$
 $k = 0 \rightarrow$ one point $(0,0,0)$

$$k \geq 0$$

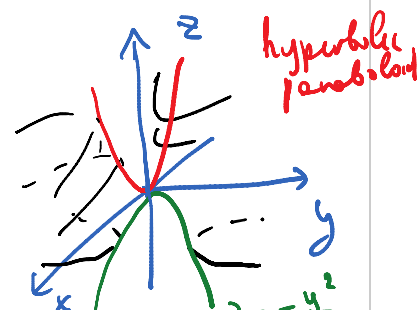
\approx

$z = k$	$k = \frac{x^2}{4} + \frac{y^2}{9} \Leftrightarrow \frac{x^2}{4k} + \frac{y^2}{9k} = 1$
$x = 0$	$z = \frac{y^2}{9} \rightarrow$ parabola in yz -plane
$y = 0$	$z = \frac{x^2}{4} \rightarrow$ parabola in xz -plane

semi axes $2\sqrt{k}$ and $3\sqrt{k}$
 $k=0 \rightarrow$ one point $(0,0,0)$
 \rightarrow both upward



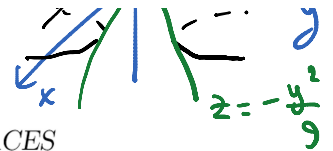
elliptic paraboloid



hyperbolic paraboloid

What is we consider : $z = \frac{x^2}{4} - \frac{y^2}{9}$:

What if we consider: $z = \frac{x^2}{4} - y^2$



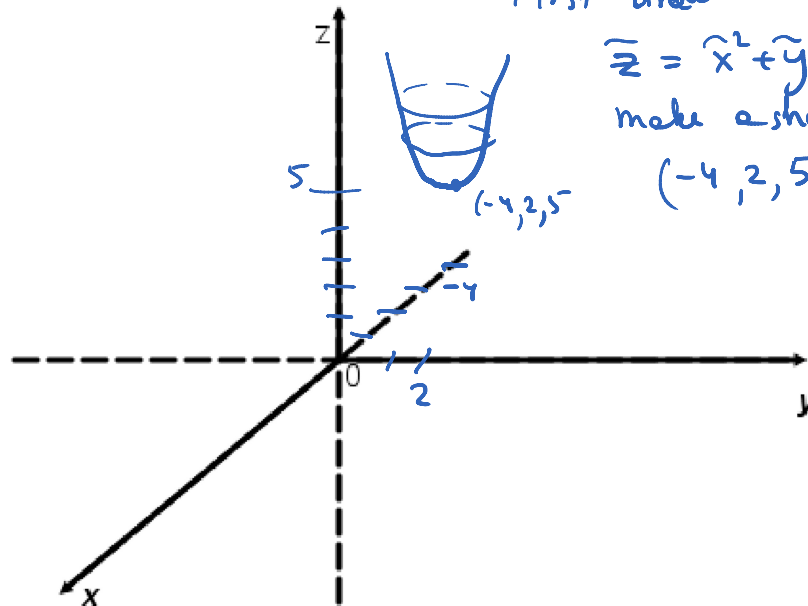
TRANSLATIONS AND REFLECTIONS OF QUADRIC SURFACES

EXAMPLE 2. Describe and sketch the surface $z = (x+4)^2 + (y-2)^2 + 5$.

$$\begin{aligned} \bar{z} - 5 &= (\bar{x} + 4)^2 + (\bar{y} - 2)^2 & x &= \bar{x} - 4 \\ y &= \bar{y} + 2 & z &= \bar{z} + 5 \end{aligned}$$

First draw

$\bar{z} = \bar{x}^2 + \bar{y}^2$ and then
make a shift by $(-4, 2, 5)$ elliptic (even circular) paraboloid

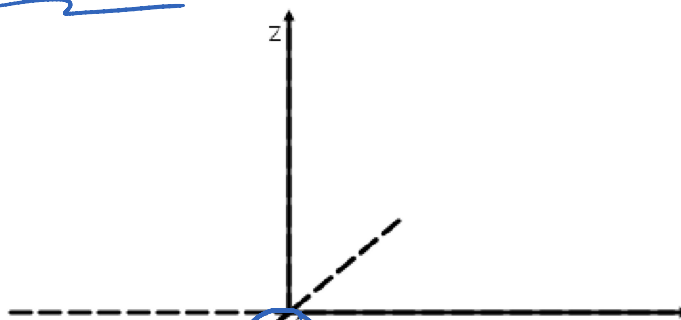


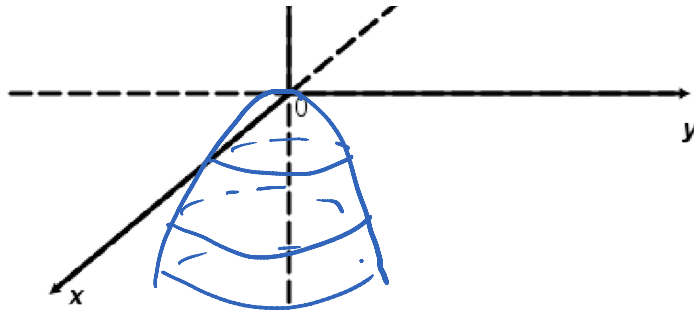
Note that replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.

EXAMPLE 3. Identify and sketch the surface.

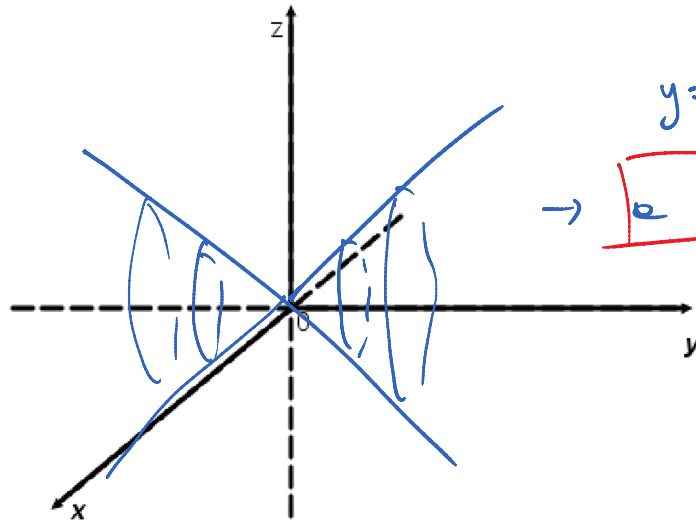
(a) $z = -(x^2 + y^2)$

replace $z = x^2 + y^2$
by $-z$





(b) $y^2 = x^2 + z^2$



yz - trace : $x=0$
 $y=z$ or $y=-z$
 $y=k \Rightarrow y^2 + z^2 = k^2 \rightarrow$
 circles

\rightarrow a cone

EXAMPLE 4. Classify and sketch the surface

$$x^2 + y^2 + z - 4x - 6y + 13 = 0.$$

Completing squares

$$x^2 - \underbrace{4x}_{2 \cdot 2} + 4 + y^2 - \underbrace{6y}_{2 \cdot 3} + 9 + z - 4 - 9 + 13 = 0$$

$$x^2 - \underbrace{4x}_{2 \cdot 2} + 4 + y^2 - \underbrace{6y}_{2 \cdot 3} + 9 + z - 4 - 9 + 13 = 0$$

$$(x-2)^2 + (y-3)^2 + z - 15 + 13 = 0$$

$$z = -(x-2)^2 - (y-3)^2 \rightarrow \text{elliptic paraboloid with vertex } (2, 3, 0) \text{ downward}$$

