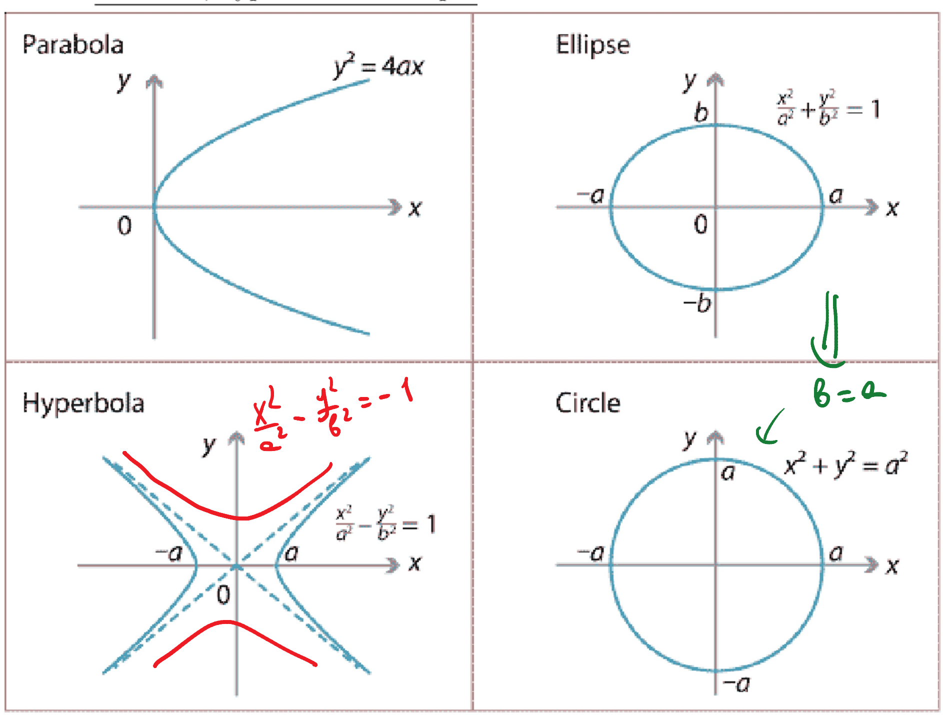
F19_LN_12_6

Sunday, September 8, 2019 8:35 PM

12.6: Quadric surfaces

REVIEW: Parabola, hyperbola and ellipse.



The most general second-degree equation in three variables x, y and z:

$$Ax^2 + By^2 + Cz^2 + axy + bxz + cyz + d_1x + d_2y + d_3z + E = 0, \qquad (1)$$

where $A, B, C, a, b, c, d_1, d_2, d_3, E$ are constants. The graph of (1) is a quadric surface.

Note if $A = B = C = a = b = c = 0$ then (1) is a linear equation and its graph is a plane (this is the case of degenerated quadric surface).

By translations and rotations (1) can be brought into one of the two standard forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

In order to sketch the graph of a surface determine the curves of intersection of the surface with planes parallel to the coordinate planes. The obtained in this way curves are called **traces** or **cross-sections** of the surface.

Quadric surfaces can be classified into 5 categories:

was discussed in sec 11.1

ellipsoids, hyperboloids, cones, paraboloids, quadric cylinders (shown in the table below.)

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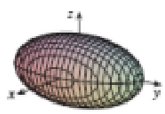
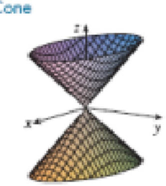
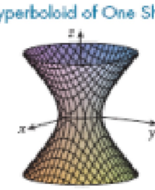
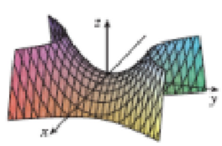
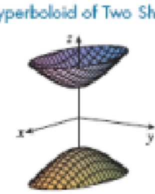
Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

The elements which characterize each of these categories:

1. Standard equation.
2. Traces (horizontal (by planes $z=k$), yz-traces (by $x=0$) and xz-traces (by $y=0$).
3. Intercepts (in some cases).

To find the equation of a trace substitute the equation of the plane into the equation of the surface.

Note, in the examples below *the constants a, b, and c are assumed to be positive.*

EXAMPLE 1. *Use traces to sketch the following quadric surfaces:*

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EXAMPLE 1. *Use traces to sketch the following quadric surfaces:*

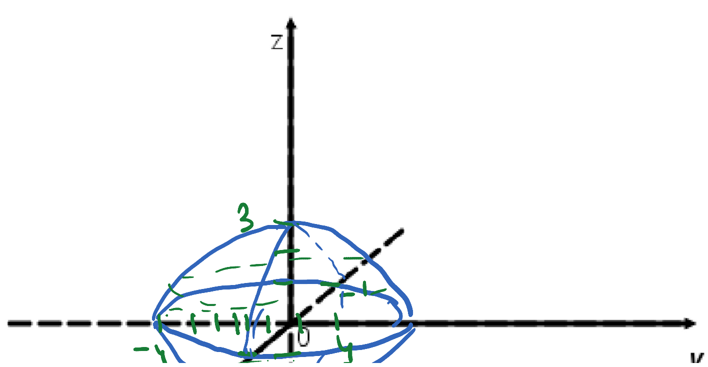
(a)

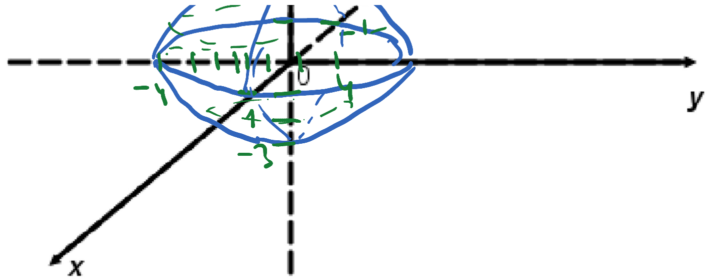
$$x^2 + \frac{y^2}{16} + \frac{z^2}{9} = 1$$

Solution

- *Find* **intercepts***:*
  - *x-intercepts: if $y = z = 0$ then* $x^2 = 1 \Rightarrow x = \pm 1$
  - *y-intercepts: if $x = z = 0$ then* $y^2 = 16 \Rightarrow y = \pm 4$
  - *z-intercepts: if $x = y = 0$ then* $z^2 = 9 \Rightarrow z = \pm 3$

- *Obtain* **traces of***:*
  - *the xy-plane: plug in $z = 0$ and get* $x^2 + \dfrac{y^2}{16} = 1$ → ellipse with semiaxis 1 and 4
  - *the yz-plane: plug in $x = 0$ and get* $\dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$ → ellipse with semiaxes 4 and 3
  - *the xz-plane: plug in $y = 0$ and get* $x^2 + \dfrac{z^2}{9} = 1$ → ellipse with semiaxes 1 and 3
  - *plug in $z = k$* → section by a horizontal plane
    $x^2 + \dfrac{y^2}{16} + \dfrac{k^2}{9} = 1 \Leftrightarrow \dfrac{x^2}{1-\frac{k^2}{9}} + \dfrac{y^2}{16\left(1-\frac{k^2}{9}\right)} = 1$ → an ellipse for $-3 < k < 3$
  - *plug in $x = k$*
  - *plug in $y = k$*





(b)

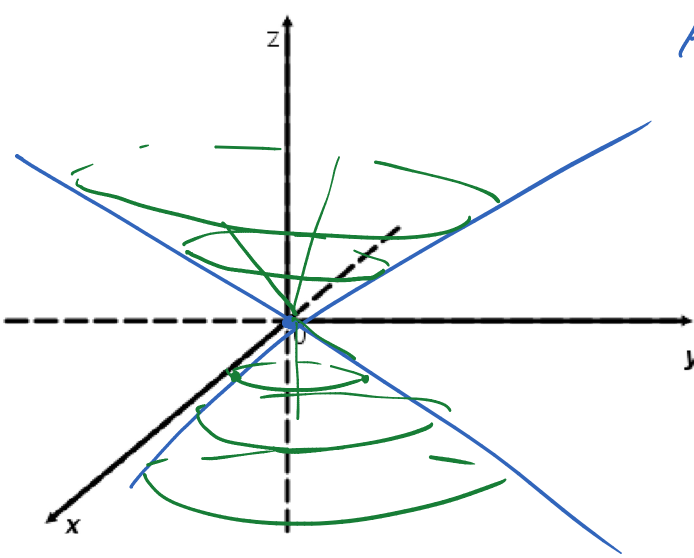
$$z^2 = x^2 + \frac{y^2}{9}$$

$z^2 - x^2 - \frac{y^2}{9} = 0$

Plane	Trace
$z = k$	$x^2 + \frac{y^2}{9} = k^2 \iff \frac{x^2}{k^2} + \frac{y^2}{9k^2} = 1 \to$ ellipse
$x = 0$	$z^2 = \frac{y^2}{9} \iff$ either $z = \frac{y}{3}$ or $z = -\frac{y}{3}$
$y = 0$	$z^2 = x^2 \iff$ either $z = x$ or $z = -x$

Using these 2 items we already can draw a surface

A cone



(c)

$$z = \frac{x^2}{4} + \frac{y^2}{9}$$

Case 1 $k = 0$: $\frac{x^2}{4} + \frac{y^2}{9} = 0 \Rightarrow$

Plane	Trace
$z = k$	$k \neq 0$

$z = \frac{x^2}{4} + \frac{y^2}{9}$

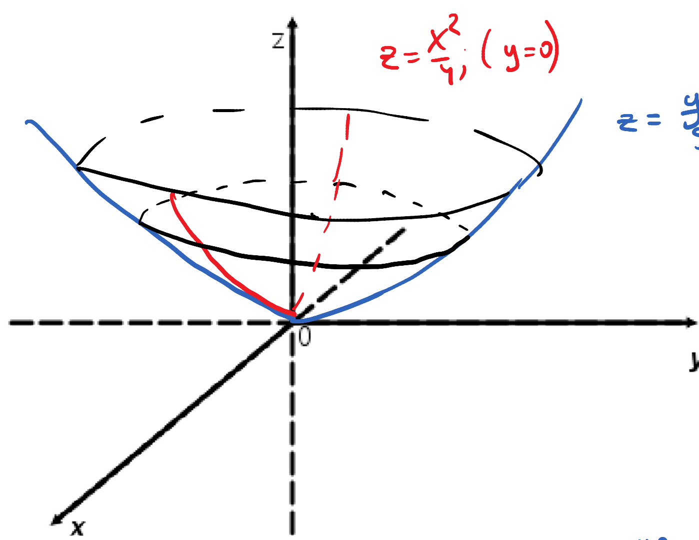
$k \geq 0 \Leftarrow$

Plane	Trace
$z = k$	$k = \frac{x^2}{4} + \frac{y^2}{9} \overset{k \neq 0}{\iff} \frac{x^2}{4k} + \frac{y^2}{9k} = 1$
$x = 0$	$z = \frac{y^2}{9} \to$ parabola
$y = 0$	$z = \frac{x^2}{4} \to$ parabola

} upward

Case 1 $k = 0$: $\frac{x^2}{4} + \frac{y^2}{9} = 0 \Rightarrow$
$x = y = 0 \to$ a point

Case 2 $k > 0 \to$ ellipse with semiaxis $2\sqrt{k}$ & $3\sqrt{k}$



elliptic paraboloid

yz-intersept: $z = -\frac{y^2}{9} \to$ parabola down

xz-intersept: ... x^2 ... y^2 ...

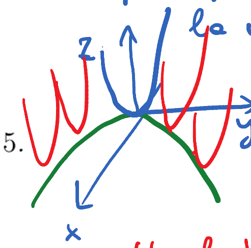
What if we consider : $z = \frac{x^2}{4} - \frac{y^2}{9}$:

yz-intersept: $z = -\frac{y^2}{9}$ → parabola down

xz-intersept: $z = \frac{x^2}{4}$ → parabola up

hyperbolic paraboloid

horizontal intercepts $z=k$ → hyperbolas if $k \neq 0$ and 2 lines if $k=0$



EXAMPLE 2. *Describe and sketch the surface* $z = (\underbrace{x+4}_{\tilde{x}})^2 + (\underbrace{y-2}_{\tilde{y}})^2 + 5.$

$\underbrace{z-5}_{\tilde{z}} = (\underbrace{x+4}_{\tilde{x}})^2 + (\underbrace{y-2}_{\tilde{y}})^2$

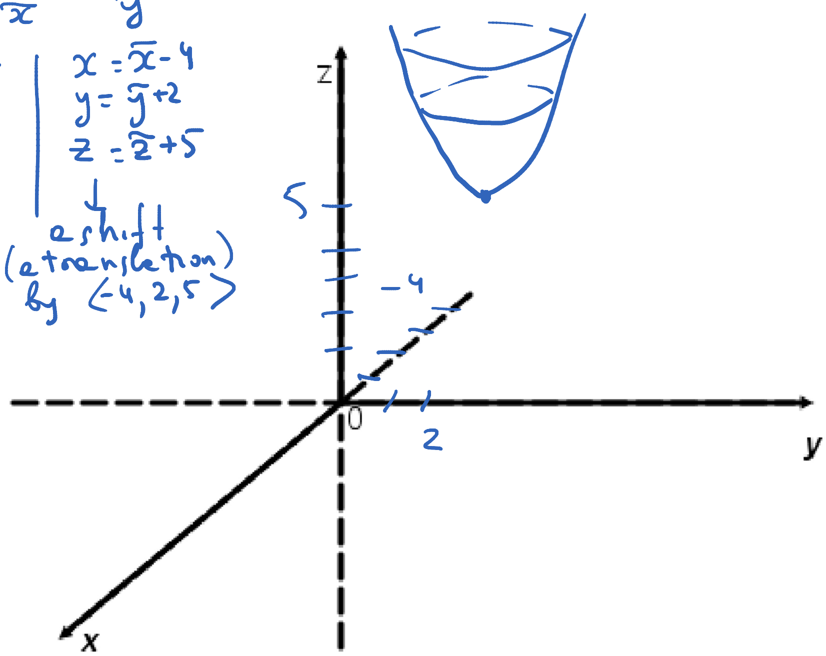
$\tilde{z} = \tilde{x}^2 + \tilde{y}^2$

elliptic (even circular) paraboloid in $\tilde{x}\tilde{y}\tilde{z}$-space

$x = \tilde{x} - 4$
$y = \tilde{y} + 2$
$z = \tilde{z} + 5$

a shift (a translation) by $\langle -4, 2, 5 \rangle$

Better look on traces of $y=k$: $z = \frac{x^2}{4} - \frac{k^2}{9}$ → parabolas up



Note that **replacing a variable by its negative in the equation of a surface causes that surface to be reflected about a coordinate plane.**

EXAMPLE 3. *Identify and sketch the surface.*

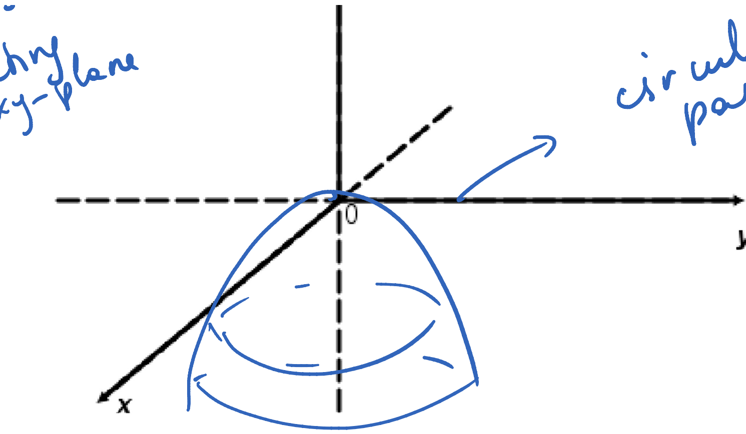
(a) $z = -(x^2 + y^2)$

We know how to draw $z = x^2 + y^2$

replacing z by $-z$ → reflecting plane

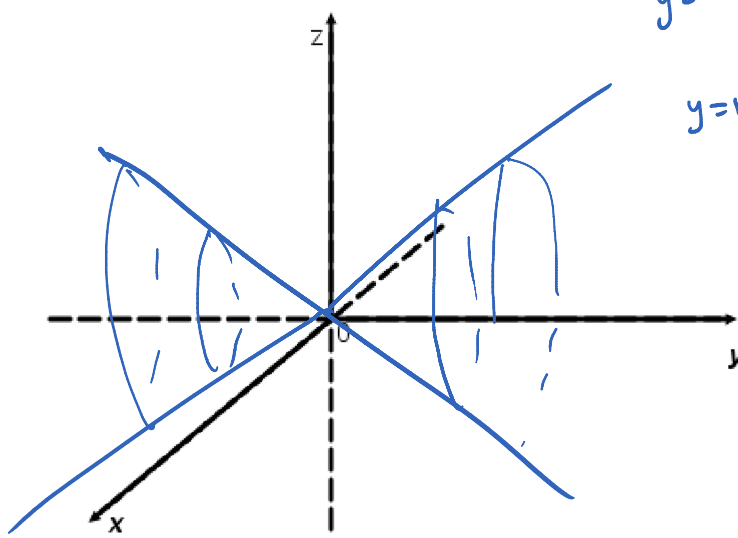
circular paraboloid ... ward

z ↓ reflecting w.r.t. xy-plane



circular paraboloid downward

(b) $y^2 = x^2 + z^2$



yz- intercept : $x=0$
$y^2 = z^2 \iff y = \pm z$
$y=k$: $x^2+y^2=k^2 \rightarrow$ circles

A cone but around y-axis

EXAMPLE 4. *Classify and sketch the surface*

$$x^2 + y^2 + z - 4x - 6y + 13 = 0.$$

Complete squares: $(x^2 - 4x + 4) - 4 + (y^2 - 6y + 9) - 9 + z + 13 = 0$

Complete squares: $(x^2 - \underbrace{4x}_{2\cdot 2} + 4) - 4 + (y^2 - \underbrace{6y}_{2\cdot 3} + 9) - 9 + z + 13 = 0$

$$(x-2)^2 + (y-3)^2 + z \cancel{-4-9+13} = 0$$

$$z = -(x-2)^2 - (y-3)^2$$

elliptic (even circular) paraboloid with vertex $(2,3,0)$ directed down.