F19_LN_13_1_13_2

Sunday, September 8, 2019 9:59 PM

13.1: Vector Functions and Space Curves

A vector function is a function that takes one or more variables and returns a vector. Let $\mathbf{r}(t)$ be a vector function whose range is a set of 3-dimensional vectors:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where $x(t), y(t), z(t)$ are functions of one variable and they are called the **component functions**.

A vector function $\mathbf{r}(t)$ is *continuous* if and only if its component functions $x(t), y(t), z(t)$ are continuous.

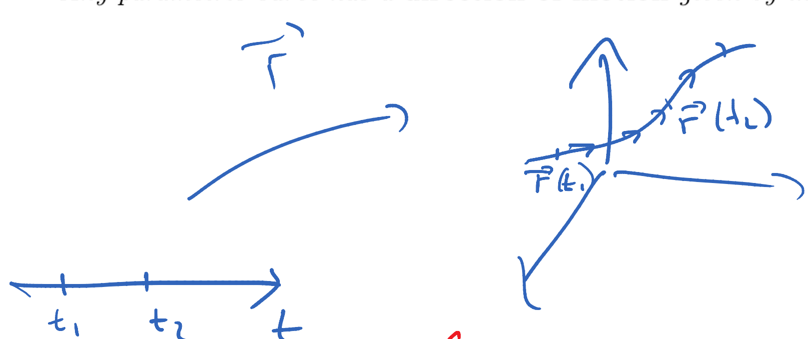
Space curve is given by **parametric equations***:*

$$C = \{(x, y, z) | x = x(t), y = y(t), z = z(t), \quad t \text{ in } I\},$$

where I is an interval and t is a **parameter***.*

FACT: Any continuous vector-function $\mathbf{r}(t)$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$.

Any parametric curve has a **direction of motion** *given by increasing of parameter.*

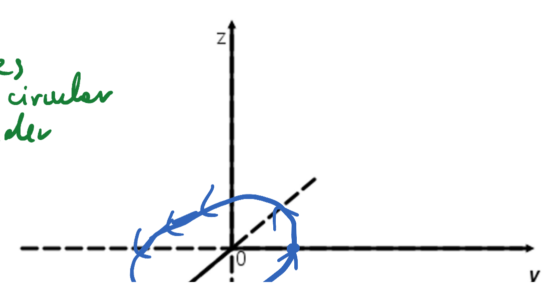


EXAMPLE 1. *Describe the curve defined by the vector function (indicate direction of motion):*

(a) $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$

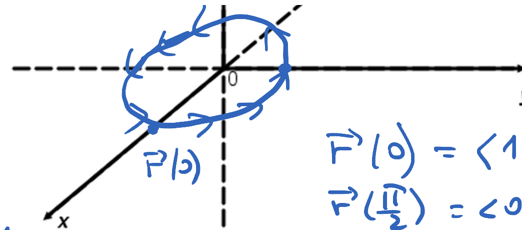
$$\begin{cases} x = \cos t \\ y = \sin t \\ z = 0 \end{cases} \qquad x^2 + y^2 = 1 \rightarrow C \text{ lies in a circular cylinder}$$

$z = 0 \Rightarrow C$ lies in xy-plane



in xy-plane

C = intersection of a cylinder with the xy-plane



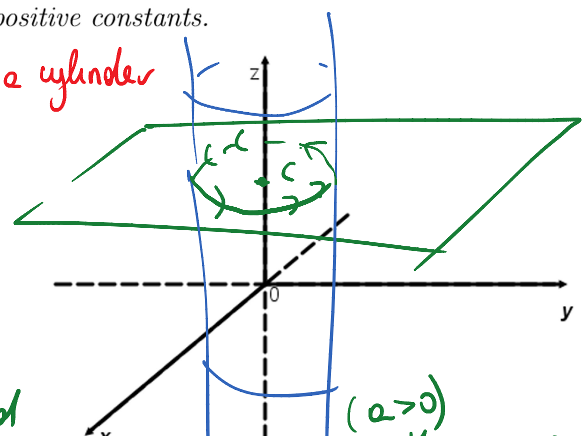
$\vec{r}(0) = \langle 1,0,0\rangle$
$\vec{r}(\frac{\pi}{2}) = \langle 0,1,0\rangle$

C is a circle in xy-plane with radius 1 around the origin. The motion is counter clockwise with unit angular velocity
(if seen from above

$x^2+y^2=\cos^2 at+\sin^2 at=1$

(b) $\mathbf{r}(t) = \langle \cos at, \sin at, c\rangle$ *where a and c are positive constants.*

$$\begin{cases} x = \cos at \\ y = \sin at \end{cases} \quad x^2+y^2=1 \to \text{a cylinder}$$
$z = c \to$ a horizontal plane

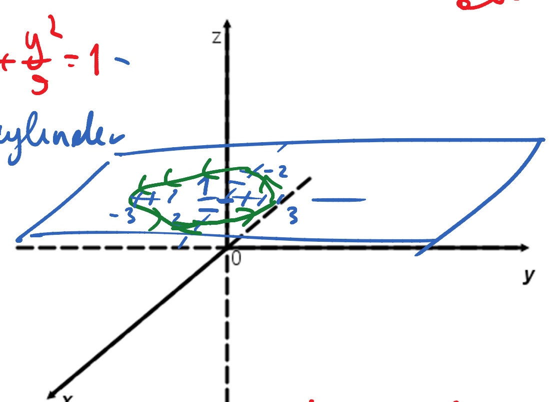


The curve is a circle of radius 1 in the horizontal plane $z=c$ around $(0,0,c)$. The motion is counter clockwise with angular velocity a $(a>0)$ if seen from above

(c) $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 1\rangle$, $0 \le t \le 2\pi$

$$\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases} \quad \left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1 \Leftrightarrow \frac{x^2}{4}+\frac{y^2}{9}=1 -$$
elliptical cylinder

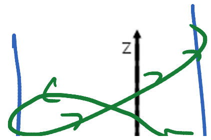
$z = 1 \to$ a horizontal plane



Ellipse in the horizontal plane and we make exactly one turn counter clockwise if seen from above

(d) $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad x^2+y^2=1$$ so the curve lies on the cylinder

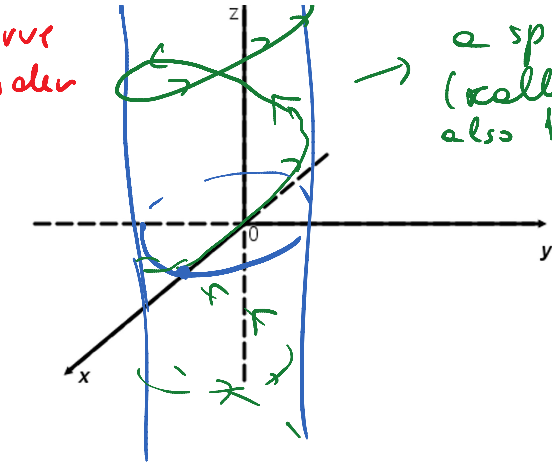


$\to$ a spiral (coil

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

$\left.\begin{matrix} x = \cos t \\ y = \sin t \end{matrix}\right\}$ $x^2+y^2=1$ so the curve lies on the cylinder

$r(0) = \langle 1, 0, 0 \rangle$

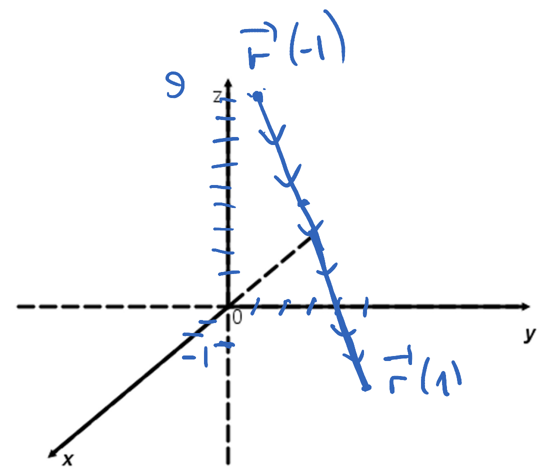


a spiral (call also helix)

(e) $\mathbf{r}(t) = \langle 1+t, 3+2t, 4-5t\rangle$, $-1 \le t \le 1$.

$$\begin{cases} x = 1+t \\ y = 3+2t \\ z = 4-5t \end{cases}$$

$\to$ a segment of straight line from $\vec{r}(-1) = \langle 0, 1, 9\rangle$ to $\vec{r}(1) = \langle 2, 5, -1\rangle$



EXAMPLE 2. *Show that the the curve given by*

$$\mathbf{r}(t) = \left\langle \sin t, 2\cos t, \sqrt{3}\sin t \right\rangle$$

lies on both a plane and a sphere. Then conclude that its graph is a circle and find its radius.

$$\begin{cases} x = \sin t \\ y = 2\cos t \\ z = \sqrt{3}\sin t \end{cases}$$

$z = \sqrt{3}x \iff \sqrt{3}x - z = 0 \to$ a plane $\Rightarrow$ our curve lies on this plane, which by the way contains the origin

$x^2+y^2+z^2 = \sin^2 t + 4\cos^2 t + 3\sin^2 t =$

$= 4\cos^2 t + 4\sin^2 t = 4 \Rightarrow$ our curve lies

$= 4\cos^2 t + 4\sin^2 t = 4$ $\Rightarrow$ our curve lies on the sphere of radius 2 around the origin

$4(\cos^2 t + \sin^2 t)$

$\Rightarrow$ our curve is the intersection of a sphere of radius 2 around the origin and a plane passing through its center

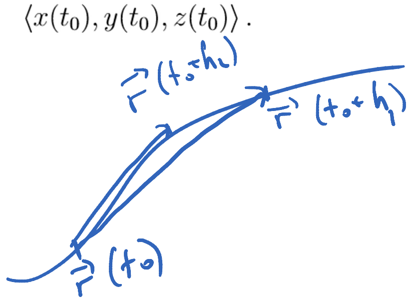
$\Rightarrow$ our curve is a circle of radius 2 with the center at the origin

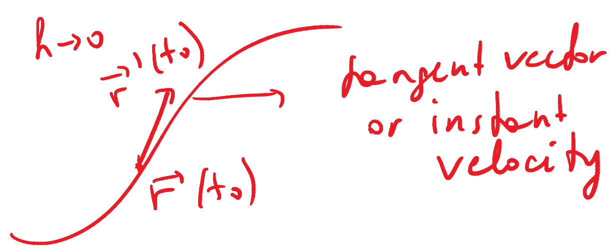
13.2 Derivatives of Vector Functions

The derivative $\mathbf{r}'$ *of a vector function* $\mathbf{r}$ *is defined just as for a real-valued function:*

$$\frac{\mathrm{d}\mathbf{r}(t_0)}{\mathrm{d}t} = \mathbf{r}'(t_0) = \lim_{h\to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h}$$

if the limit exists. The derivative $\mathbf{r}'(t_0)$ *is the* tangent vector *to the curve* $\mathbf{r}(t)$ *at the point* $\mathbf{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$.





THEOREM 3. *If the functions* $x(t), y(t), z(t)$ *are differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

EXAMPLE 4. *Given* $\mathbf{r}(t) = (1+t)^2\mathbf{i} + e^t\mathbf{j} + \sin 3t\mathbf{k}$.

(a) *Find* $\mathbf{r}'(t)$ $= \langle ((1+t)^2)', (e^t)', (\sin 3t)' \rangle = \boxed{\langle 2(1+t), e^t, 3\cos 3t\rangle}$

(b) *Find a tangent vector to the curve at* $t = 0$.

tangent vector $= \vec{r}\,'(0) = \langle 2, 1, 3\rangle$

(c) *Find a tangent line to the curve at* $t = 0$.

This is a line passing through $\vec{r}(0)$ in the direction of $\vec{r}\,'(0)$. $\vec{r}(0) = \langle 1, e^0, \sin 0\rangle = \langle 1, 1, 0\rangle \Rightarrow$

The parametric equation is $x = 1 + 2t,\ y = 1 + t,\ z = 3t$

(c′) *Find a tangent line to the curve at the point* $(1, 1, 0)$.

Find parameter t s.t $\vec{r}(t) = (1, 1, 0) \Leftrightarrow \begin{cases} (1+t)^2 = 1 \\ e^t = 1 \Rightarrow t = 0 \\ \sin 3t = 0 \end{cases}$

Then we reduced to (c)