F19_LN_13_1_13_2

Sunday, September 8, 2019 10:00 PM

13.1: Vector Functions and Space Curves

A vector function is a function that takes one or more variables and returns a vector. Let $\mathbf{r}(t)$ be a vector function whose range is a set of 3-dimensional vectors:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

where $x(t), y(t), z(t)$ are functions of one variable and they are called the **component functions**.

A vector function $\mathbf{r}(t)$ is *continuous* if and only if its component functions $x(t), y(t), z(t)$ are continuous.

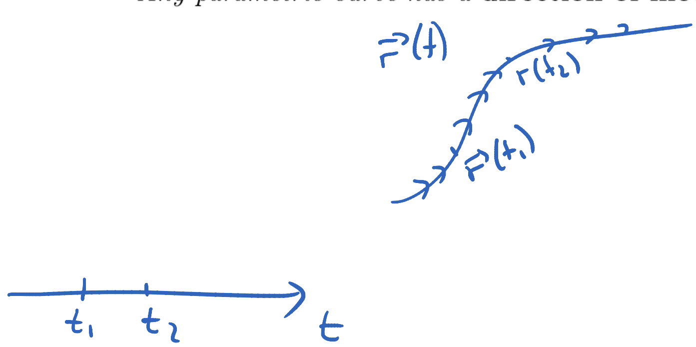
Space curve is given by **parametric equations***:*

$$C = \{(x, y, z) | x = x(t), y = y(t), z = z(t), \quad t \text{ in } I\},$$

where I is an interval and t is a **parameter***.*

FACT: Any continuous vector-function $\mathbf{r}(t)$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$.

Any parametric curve has a **direction of motion** *given by increasing of parameter.*

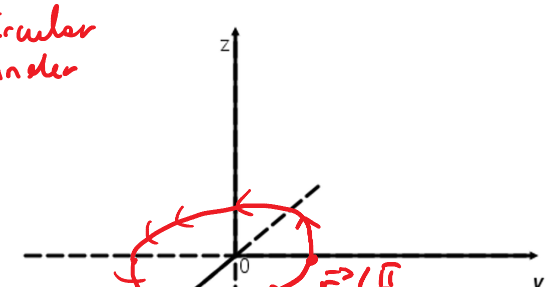


EXAMPLE 1. *Describe the curve defined by the vector function (indicate direction of motion):*

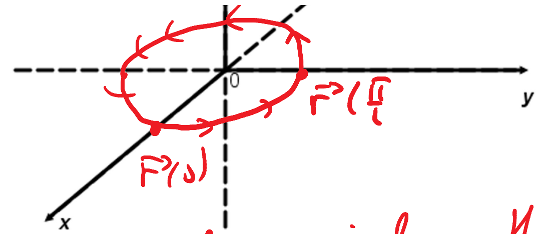
(a) $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$

$$\left\{\begin{array}{l} \left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \Rightarrow x^2+y^2=1 \rightarrow \text{a circular cylinder} \\ z = 0 \rightarrow xy\text{-plane} \end{array}\right.$$

Intersection of a cylinder



Intersection of a cylinder with the xy-plane is a circle:



Direction of motion: $\vec r(0) = \langle 1, 0, 0\rangle$, $\vec r(\frac{\pi}{2}) = \langle 0, 1, 0\rangle$ $\rightarrow$ we move along a circle with unit angular velocity

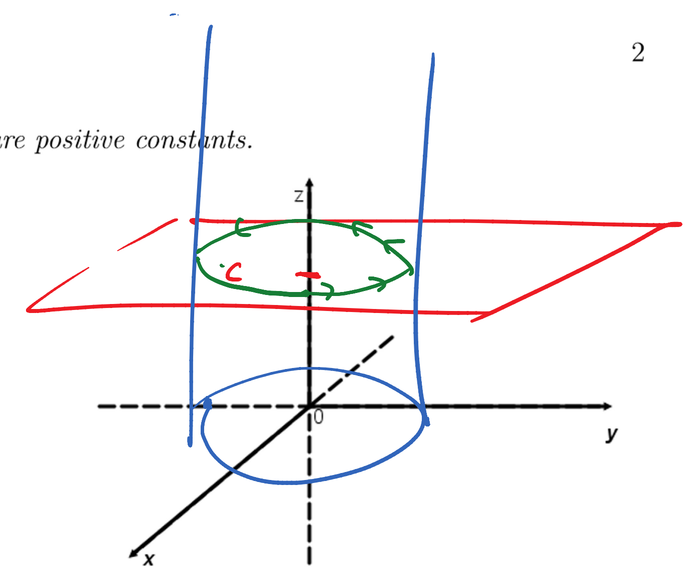
(b) $\mathbf{r}(t) = \langle \cos at, \sin at, c\rangle$ *where* a *and* c *are positive constants.*

$$\left.\begin{array}{l} x = \cos at \\ y = \sin at \end{array}\right\} \Rightarrow x^2+y^2=1$$

$z = c \Rightarrow$ in a horizontal plane

$\downarrow$

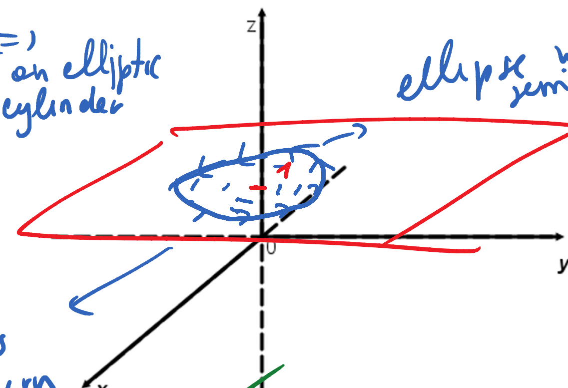
A unit circle in the plane $z=c$. We move with angular velocity a



(c) $\mathbf{r}(t) = \langle 2\cos t, 3\sin t, 1\rangle$, $0 \le t \le 2\pi$

$$\left\{\begin{array}{l} \left.\begin{array}{l} x = 2\cos t \\ y = 3\sin t \end{array}\right\} \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \iff \frac{x^2}{4}+\frac{y^2}{9} = 1 \rightarrow \text{an elliptic cylinder} \\ z = 1 \rightarrow \text{a horizontal plane } z=1 \end{array}\right.$$

ellipse with semi-axis 2 and 3

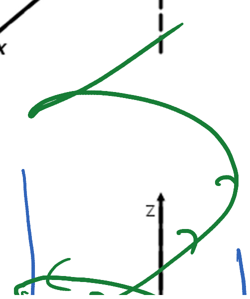


makes one turn

(d) $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$

$$\left\{\begin{array}{l} \left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} x^2+y^2=1 \Rightarrow \end{array}\right.$$

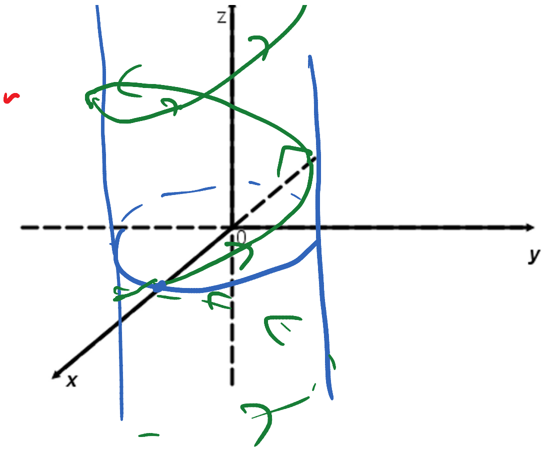
a curve lies in a cylinder



$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

$x^2+y^2=1 \Rightarrow$ a curve lies in a cylinder

the motion with constant velocity of z-component → spiral



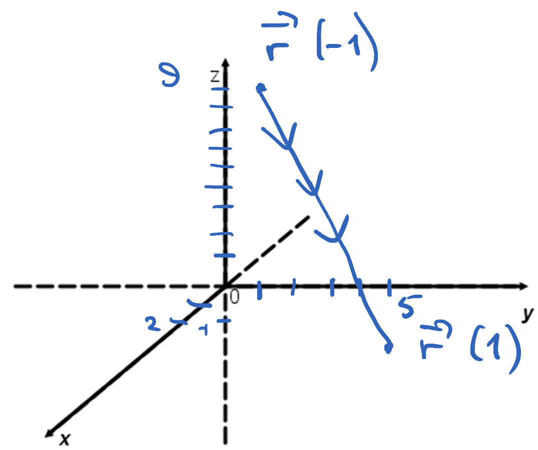
(e) $\mathbf{r}(t) = \langle 1+t, 3+2t, 4-5t\rangle$, $-1 \le t \le 1$.

$$\begin{cases} x = 1+t \\ y = 3+2t \\ z = 4-5t \end{cases}$$

the segment of the straight line connecting $\vec{r}(-1)$ & $\vec{r}(1)$

$\vec{r}(-1) = (0,1,9)$

$\vec{r}(1) = (2,5,-1)$



EXAMPLE 2. *Show that the the curve given by*

$$\mathbf{r}(t) = \left\langle \sin t, 2\cos t, \sqrt{3}\sin t \right\rangle$$

lies on both a plane and a sphere. Then conclude that its graph is a circle and find its radius.

$$\begin{cases} x = \sin t \\ y = 2\cos t \\ z = \sqrt{3}\sin t \end{cases}$$

$z = \sqrt{3}x \Leftrightarrow \sqrt{3}x - z = 0$ – a plane passing through the origin

our curve lies on

$x^2+y^2+z^2 = \sin^2 t + 4\cos^2 t + 3\sin^2 t =$

$= 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4 \Rightarrow$

$x^2+y^2+z^2 = \sin^2 t + 4\cos^2 t + 3\sin^2 t$

$= 4\cos^2 t + 4\sin^2 t = 4(\underbrace{\cos^2 t + \sin^2 t}_{1}) = 4 \Rightarrow$

$x^2+y^2+z^2 = 4 \Rightarrow$ our curve lies on a sphere of radius 2 around the origin

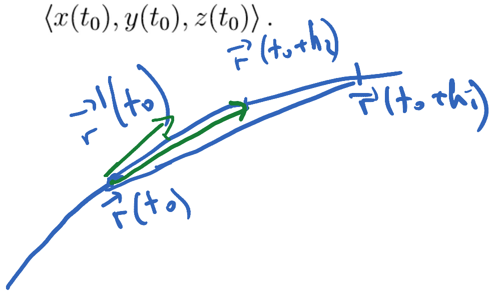
$\Rightarrow$ our curve is a circle of radius 2 with the center at the origin.

13.2 Derivatives of Vector Functions

The derivative $\mathbf{r}'$ *of a vector function* $\mathbf{r}$ *is defined just as for a real-valued function:*

$$\frac{\mathrm{d}\mathbf{r}(t_0)}{\mathrm{d}t} = \mathbf{r}'(t_0) = \lim_{h\to 0} \underbrace{\frac{\overbrace{\mathbf{r}(t_0+h)-\mathbf{r}(t_0)}^{\to \text{ displacement}}}{h}}_{\text{an average velocity on a segment } [t_0,\, t_0+h]}$$

if the limit exists. The derivative $\mathbf{r}'(t_0)$ *is the* tangent vector *to the curve* $\mathbf{r}(t)$ *at the point* $\mathbf{r}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle$ *.*



THEOREM 3. *If the functions* $x(t), y(t), z(t)$ *are differentiable, then*

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

EXAMPLE 4. *Given* $\mathbf{r}(t) = (1+t)^2\mathbf{i} + e^t\mathbf{j} + \sin 3t\mathbf{k}$.

(a) *Find* $\mathbf{r}'(t)$ $= \langle ((1+t)^2)', (e^t)', (\sin 3t)' \rangle = \boxed{\langle 2(1+t), e^t, 3\cos 3t\rangle}$

(b) *Find a tangent vector to the curve at* $t = 0$.

$r'(0) = \langle 2(1+0), e^0, 3\cos(3\cdot 0)\rangle = \boxed{\langle 2, 1, 3\rangle}$

(c) *Find a tangent line to the curve at* $t = 0$.

The tangent line is the line through $\vec{r}(0)$ in the direction $\vec{r}\,'(0)$. $\vec{r}(0) = \langle 1+0, e^0, \sin(3\cdot 0)\rangle = (1, 1, 0)$

The parametric equation of the tangent line is: $\begin{cases} x = 1+2t \\ y = 1+t \\ z = 3t \end{cases}$

(c′) *Find a tangent line to the curve at the point* $(1, 1, 0)$.

Find the parameter t such that

$\vec{r}(t) = (1,1,0) \iff \begin{cases} (1+t)^2 = 1 \\ e^t = 1 \\ \sin 3t = 0 \end{cases} \Rightarrow t = 0 \Rightarrow$

we have a tangent line to the curve at $t=0$ that was done in (c)