F19_LN_13_3

Thursday, September 12, 2019 9:32 PM

13.3: Arc Length and curvature of a curve

The length of a curve

The length of a plane curve with parametric equations $x = x(t)$, $y = y(t)$, $a \le t \le b$, as the limit of lengths of inscribed broken lines is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

under the assumption that $x'(t)$ and $y'(t)$ exist and continuous

In exactly the same way the length of a space curve given by the parametric equation

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

as the limit of lengths of inscribed broken lines, is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$$

under the assumption that $x'(t)$, $y'(t)$, and $z'(t)$ exist and continuous.

$\Delta t \to 0$

$$\sum \Delta L = \sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\, \Delta t$$

Both cases of lengths of plane and space curves can be simultaneously written as

$$L = \int_a^b |\mathbf{r}'(t)|\, dt$$

where

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is the vector function describing the curve.

EXAMPLE 1. *Find the length of the curve:*

(a) $\mathbf{r}(t) = \langle -2t, 5\cos t, 5\sin t \rangle, \quad -10 < t < 10.$

$\mathbf{r}'(t) = \langle -2, -5\sin t, 5\cos t \rangle \qquad |\mathbf{r}'(t)| = \sqrt{4 + 25\sin^2 t + 25\cos^2 t}$

(a) $\mathbf{r}(t) = \langle -2t, 5\cos t, 5\sin t\rangle, \quad -10 < t < 10.$

$$r'(t) = \langle -2, -5\sin t, 5\cos t\rangle, \quad |r'(t)| = \sqrt{4+\underbrace{25\sin^2 t + 25\cos^2 t}_{25(\underbrace{\cos^2 t+\sin^2 t}_{1})=25}}$$

$$= \sqrt{4+25} = \sqrt{29}$$

$$L = \int_{-10}^{10} |r'(t)|\,dt = \int_{-10}^{10} \sqrt{29}\,dt = \sqrt{29}\,t\Big|_{-10}^{10} = 20\sqrt{29}$$

(b) $\mathbf{r}(t) = \langle 4t, \frac{2\sqrt{2}}{3}t^3, \frac{1}{5}t^5\rangle, \quad 0 \le t \le 2$

$$r'(t) = \langle 4, 2\sqrt{2}\,t^2, t^4\rangle, \quad |r'(t)| = \sqrt{\underbrace{16+8t^4+t^8}_{(t^4+4)^2}} = t^4+4$$

$$L = \int_0^2 |r'(t)|\,dt = \int_0^2 (t^4+4)\,dt = \frac{t^5}{5}+4t\Big|_0^2 = \frac{2^5}{5}+8 = \frac{32}{5}+8 = \boxed{\frac{72}{5} = 14.4}$$

The Arc Length Function/Parametrization

Assume again that a curve C is given by the vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

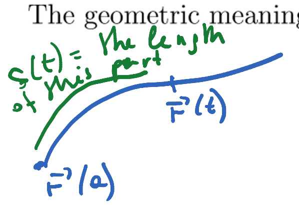
The function

$$s(t) = \int_a^t |\mathbf{r}'(\tau)|\,d\tau = \int_a^t \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\,d\tau \Leftrightarrow \frac{ds}{dt} = |\mathbf{r}'(t)|, \ s(a) = 0.$$

$$\Downarrow \quad \frac{dt}{ds} = \frac{1}{|r'(t)|}$$

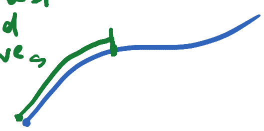
is called the *arc length function* of the curve C.

The geometric meaning: $s(t)$ is the length of the part of C between the points $\mathbf{r}(a)$ and $\mathbf{r}(t)$.



If L is the length of the curve C then for every $s \in [0, L]$ there exists unique $t(s)$ such that the length of the curve between $\mathbf{r}(a)$ and $\mathbf{r}\big(t(s)\big)$ is equal to s.

Take a thread of length s and put it on a curve, with one endpoint at $\vec{r}(a)$



The other endpoint of the thread will be $\vec{r}(t)$ for some t that you call $t(s)$

The curve given by the vector function

with ... at $\vec{r}'(a)$

The curve given by the vector function

$$\widetilde{\mathbf{r}}(s) := \mathbf{r}\big(t(s)\big)$$

trace out the same curve C in $\mathbb{R}^3$, as $\mathbf{r}(t)$, but the motion along C with respect to the new parameter s is different ~~, than~~ from the motion with respect to the old parameter t. More precisely, the motion along C with respect to s has the unit speed at every points, because

Chain rule

$$\left|\frac{d\widetilde{\mathbf{r}}}{ds}\right| = \left|\frac{d\mathbf{r}}{dt}\right|\underbrace{\left|\frac{dt}{ds}\right|}_{\frac{1}{\left|\frac{dr}{dt}\right|}} = 1.$$

By finding the function $t = t(s)$ (i.e., the inverse to the arc length function), one *reparametrizes the curve C with respect to the arc length*

EXAMPLE 2. *Reparametrize the curve with respect to arc length measured from the point where $t = 0$ in the direction of increasing t, i.e. find $t(s)$, if*

$\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (1 - 3t)\mathbf{j} + (1 + 2t)\mathbf{k}$

$r'(t) = \langle 4, -3, 2 \rangle$

1. Find the arc length function $s(t)$:

$$s(t) = \int_0^t |r'(\tau)|\,d\tau = \int_0^t \sqrt{4^2 + (-3)^2 + 2^2}\,d\tau = \int_0^t \sqrt{16+9+4}\,d\tau = \sqrt{29}\,t$$

$$s = \sqrt{29}\,t \Rightarrow \boxed{t = \frac{s}{\sqrt{29}}} \Rightarrow \underbrace{\vec{\tilde{r}}(s) = \left(1 + \frac{4s}{\sqrt{29}}\right)\hat{i} + \left(1 - \frac{3s}{\sqrt{29}}\right)\hat{j}}_{} + \left(1 + \frac{2s}{\sqrt{29}}\right)\hat{k}$$

the same line but we move along it with unit speed

Curvature of the curve

The *unit tangent vector*

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Assume that the curve is parametrized by arc length parameter s, i.e. is represented by a vector function $\mathbf{r}(s)$ such that $|r'(s)| = 1$. Then the curvature $k(s)$ (at the point $\mathbf{r}(s)$ is defined as follows

$$k(s) = |r''(s)| = \left|\frac{d\mathbf{T}}{ds}\right|,$$

the magnitude of the acceleration if we move along the curve with the unit speed

i.e., the curvature of the curve is the magnitude of the acceleration if one moves along the curve with unit speed.

If a curve C is parametrized by a parameter T (which not necessary by the arc length parametriation), then

$$\boxed{k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}} \qquad (1)$$

$$k = \left|\frac{d\vec{T}}{ds}\right| \underset{\text{chain rule}}{=} \left|\frac{d\vec{T}}{dt}\right| \cdot \underbrace{\left|\frac{dt}{ds}\right|} = \left|\frac{d\vec{T}}{dt}\right| \frac{1}{|r'(t)|}$$

EXAMPLE 3. *What is the curvature of the circle of radius R.*

Use parametric equation $x = R\cos t$, $y = R\sin t$

$r'(t) = \langle -R\sin t, R\cos t\rangle \Rightarrow T(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \langle -\sin t, \cos t\rangle$

$|r'(t)| = R$

$T'(t) = \langle -\cos t, -\sin t\rangle \Rightarrow |T'(t)| = 1$

$k(t) = \frac{|T'(t)|}{|r'(t)|} = \boxed{\frac{1}{R}}$

REMARK 4. The curvature of a curve is identically equal to zero if and only if the curve is a straight line

EXAMPLE 5. *Let* $\mathbf{r}(t) = \langle 3\cos 2t, 3\sin 2t, -3t\rangle$, $t \in \mathbb{R}$.

(a) *Find the unit tangent $T(t)$ of the curve given by $\mathbf{r}(t)$*

$r'(t) = \langle -6\sin 2t, 6\cos 2t, -3\rangle$, $|r'(t)| = \sqrt{\underbrace{36\sin^2 2t + 36\cos^2 2t}_{36} + 9} =$

$= \sqrt{\underbrace{45}_{5\cdot 9}} = 3\sqrt{5}$

$T(t) = \frac{r'(t)}{|r'(t)|} =$

$= \frac{1}{\cancel{3}\sqrt{5}} \cdot \cancel{3}\langle -2\sin 2t, 2\cos 2t, -1\rangle =$

$= \boxed{\frac{1}{\sqrt{5}}\langle -2\sin 2t, 2\cos 2t, -1\rangle}$

(b) *Find the curvature of the curve given by $\mathbf{r}(t)$.*

$k(t) = \frac{|T'(t)|}{|r'(t)|}$

$T'(t) = \frac{1}{\sqrt{5}}\langle -4\cos 2t, -4\sin 2t, 0\rangle = -\frac{4}{\sqrt{5}}\underbrace{\langle \cos 2t, \sin 2t, 0\rangle}_{\text{unit}}$

$\Rightarrow |T'(t)| = \frac{4}{\sqrt{5}} \Rightarrow k(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{4}{\sqrt{5}}}{3\sqrt{5}} = \frac{4}{3\cdot 5} = \frac{4}{15}$

$\Rightarrow |T'(t)| = \frac{4}{\sqrt{5}} \Rightarrow k(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{4}{\sqrt{5}}}{3\sqrt{5}} = \frac{4}{3 \cdot 5} = \frac{4}{15}$ unit

REMARK 6. From (1), using the fact that given two vctors **a** and **b** one has $\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2} = |\mathbf{a}\times\mathbf{b}|$, one can deduce the following formula for the curvature in terms of $\mathbf{r}(\mathbf{t})$ and their derivatives up to order 2:

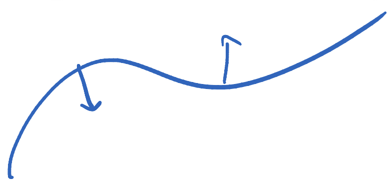
$$k(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Prncipal unit normal vector

Note that $|\mathbf{T}|^2 = \mathbf{T}(t)\cdot\mathbf{T}(t) = 1$. Differentiating this identity we get that $2\mathbf{T}'(t)\cdot\mathbf{T}(t) = 0$, i.e $\mathbf{T}'(t)\perp\mathbf{T}(t)$. Assume that $\mathbf{T}'(t)\neq 0$ (equivalently the curvature $k(t)\neq 0$). Normalizing $T'(t)$ we obtain so called *principal unit normal vector* or simply *unit normal*:

$$\mathbf{N}(t) := \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Geometric meaning of the unit normal: the direction of the acceleration if one moves with along the curve with the unit speed.



EXAMPLE 7. *Find the unit normal for the curve from Example 5.*

$N(t) = \frac{1}{|T'(t)|} T'(t) = -\langle \cos 2t, \sin 2t, 0\rangle$

REMARK 8 (The notions discussed here will not be used in homework or tests, it is just an extra material). The vector

$$\mathbf{B}(T) := \mathbf{T}(t)\times\mathbf{N}(t)$$

is called the *binormal vector*. Using the binormal vector one can define another function $\tau(t)$ on the curve called *torsion*. In the arc length parameter s

$$|\tau(s)| = \left|\frac{d\mathbf{B}(s)}{ds}\right|.$$

In the arbitrary parameter T

$$|\tau(s)| = \left|\frac{\quad}{ds}\right|.$$

In the arbitrary parameter T

$$|\tau(t)| = \frac{|B'(t)|}{|\mathbf{r}'(t)|}$$

The triple of vectors $\big(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\big)$ is called the *Frenet-Serret moving frame* and it is the basic tool in the study of curves in $\mathbb{R}^3$ up to the rigid motions, i.e. translations and rotations. Using the Frenet-Serres it can be shown that the curvature and the torsion define the curve in $\mathbb{R}^3$ uniquely up to a rigid motion.

The torsion $\tau(t) \equiv 0$ (for every t) $\Longleftrightarrow$ the curve lies in a plane