F19_LN_14_1

Tuesday, September 17, 2019 8:48 PM



# 14.1: Functions of Several Variables

Consider the following formulas:

$$z = 2 - x - 4y \tag{1}$$

$$z = x^2 + y^2 \tag{2}$$

$$z = \sqrt{x^2 + y^2} \tag{3}$$

$$z = \sqrt{1 - x^2 - y^2} \tag{4}$$

| ordered pair of independent variables $(x, y)$ (input) | $\longrightarrow$ | $f(x,y)$ | $\longrightarrow$ | dependent variable $z$ (output) |
|---|---|---|---|---|

DEFINITION 1. *Let $D \subset \mathbb{R}^2$. A* **function** *$f$* **of two variables** *is a rule that assigns to each ordered pair $(x, y)$ in $D$ a unique real number denoted by $f(x, y)$.*

*The set $D$ is the* **domain** *of $f$ and the* **range** *of $f$ is the set of values that $f$ takes on, that is $\{f(x, y)|(x, y) \in D\}$.*

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, $x = f(y, z)$.

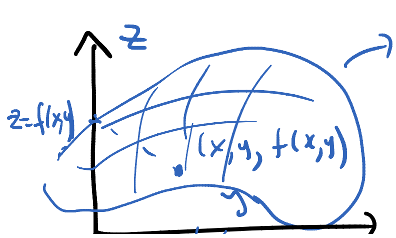
- **GRAPH of** $f(x, y)$.

  Recall that a graph of a function $f$ of one variable is a curve $C$ with equation $y = f(x)$.

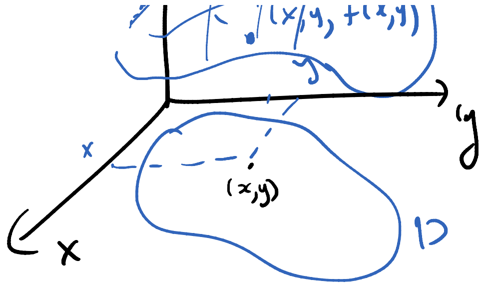
DEFINITION 3. *The* **graph** *of $f$ with domain $D$ is the set:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 | z = f(x, y). \ \ (x, y) \in D\right\}.$$

The graph of a function $f$ of two variables is a surface $S$ in three dimensional space with equation $z = f(x, y)$.




graph is a surface






EXAMPLE 4. *Find the domain and sketch the graph of the functions (1)-(4). What is the range?*

(1) $\underset{f(x,y)}{z} = 2 - x - 4y$

$D = \mathbb{R}^2$

range $= \mathbb{R}$

(for given $z$ take
$x = 2 - z$ and $y = 0$
$f(2-z, 0) = 2 - (2-z) = z$)

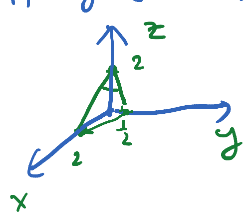
The graph is a plane and so I need 3 point to determine it.

$z = 2 - x - 4y \Leftrightarrow x + 4y + z = 2$

If $x = y = 0 \Rightarrow z = 2 \Rightarrow (0,0,2)$ is in the graph

If $x = z = 0 \Rightarrow 4y = 2 \Rightarrow y = \frac{1}{2} \Rightarrow (0, \frac{1}{2}, 0)$ is in the graph

If $y = z = 0 \Rightarrow x = 2 \Rightarrow (2,0,0)$ is in the graph



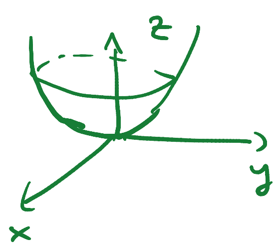

<u>Rem</u> Alternatively, the graph is the plane through $(0,0,2)$ orthogonal to the vector $\langle 1,4,1 \rangle$

(2) $z = x^2 + y^2$

$D = \mathbb{R}^2$

range : $x^2 + y^2 \geq 0 \Rightarrow$ range $= \{z : z \geq 0\} = \mathbb{R}_{\geq 0}$

The graph is a circular paraboloid




(3) $z = \sqrt{x^2 + y^2}$

$D = \mathbb{R}^2$

range $= \mathbb{R}_{\geq 0}$

square it $z^2 = x^2 + y^2$ &

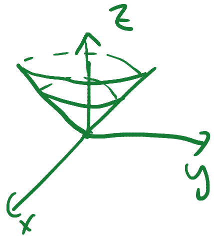
(4) $z = \sqrt{1 - x^2 - y^2} \Rightarrow 1 - x^2 - y^2 \geq 0 \Rightarrow$

$D = \{(x,y) : x^2 + y^2 \leq 1\}$, i.e. the unit disk around the origin, including the boundary circle

range = $\mathbb{R}_{\geq 0}$

$z=\sqrt{x^2+y^2} \overset{\text{square it}}{\iff} z^2=x^2+y^2$ &

$z\geq 0 \Rightarrow$ a part of the cone with $z\geq 0$

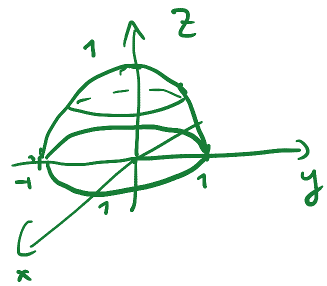



disk around the origin, including the boundary circle

Graph:

$z=\sqrt{1-x^2-y^2} \overset{\text{square it}}{\iff} z^2=1-x^2-y^2$ & $z\geq 0$

$\iff \begin{cases} x^2+y^2+z^2=1 \\ z\geq 0 \end{cases} \to$ upper hemisphere of radius 1 around $(0,0,0)$

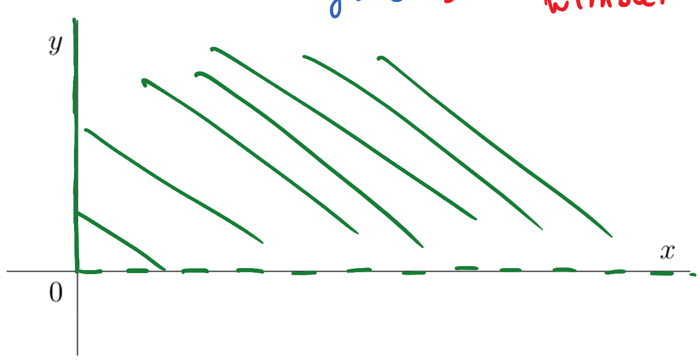



Range : $\{z : 0\leq z\leq 1\}$



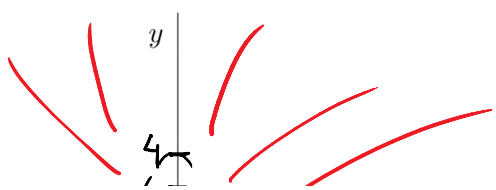
EXAMPLE 5. *Sketch the domain of each of the following:*

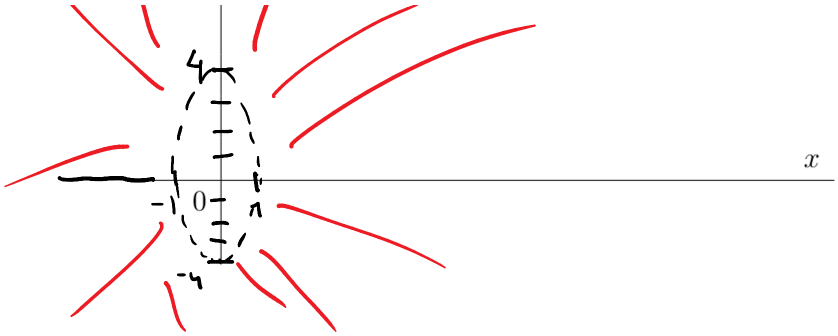
**(a)** $z=\sqrt{x}-\dfrac{5}{\sqrt{y}}$ $\Rightarrow$ $\left.\begin{matrix} x\geq 0 \\ y>0 \end{matrix}\right\}$ the first quadrant without the x-axis




**(b)** $z=\ln(x^2+\dfrac{y^2}{16}-1)$ $\Rightarrow x^2+\dfrac{y^2}{16}-1>0 \iff x^2+\dfrac{y^2}{16}>1$

$D=\{(x,y): x^2+\frac{y^2}{16}>1\}$ - the region outside of the ellipse






the region outside of the ellipse $x^2+\frac{y^2}{16}=1$ (not including the ellipse)

• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

DEFINITION 6. *The* **level (contour) curves** *of a function of two variables are the curves with equations*

$$f(x, y) = k,$$

*where $k$ is a constant in the range of $f$.*

A level curve is the locus of all points at which $f$ takes a given value $k$ ( it shows where the graph of $f$ has height $k$).



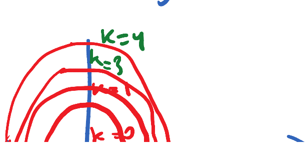
EXAMPLE 7. *Sketch the level curves of the functions (2) and (3) for the values $k = 0, 1, 2, 3, 4$:*

(2) $z = x^2 + y^2$

$x^2+y^2=k$ $(k \geq 0)$

a circle of radius $\sqrt{k}$

| k | equation | radius |
|---|---|---|
| $k=0$ | $(0,0)$ | $0$ |
| $k=1$ | $x^2+y^2=1$ | $1$ |
| $k=2$ | $x^2+y^2=2$ | $\sqrt{2}$ |
| $k=3$ | $x^2+y^2=3$ | $\sqrt{3}$ |
| $k=4$ | $x^2+y^2=4$ | $2$ |

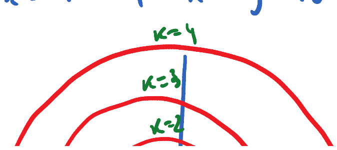


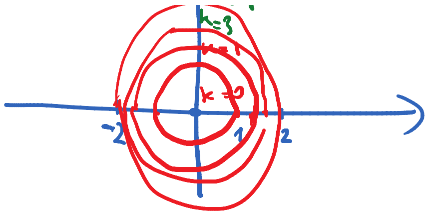

(3) $z = \sqrt{x^2 + y^2}$

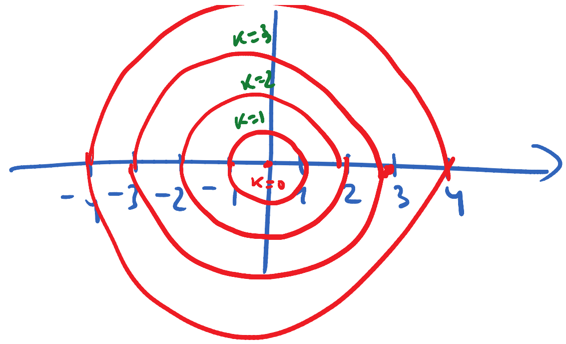
$\sqrt{x^2+y^2}=k \Rightarrow x^2+y^2=k^2$

a circle of radius $k$

| k | equation | radius |
|---|---|---|
| $k=0$ | $(0,0)$ | $0$ |
| $k=1$ | $x^2+y^2=1$ | $1$ |
| $k=2$ | $x^2+y^2=4$ | $2$ |
| $k=3$ | $x^2+y^2=9$ | $3$ |
| $k=4$ | $x^2+y^2=16$ | $4$ |






- **Functions of three variables.**

DEFINITION 8. *Let $D \subset \mathbb{R}^3$. A* **function** $f$ **of three variables** *is a rule that assigns to each ordered pair $(x, y, z)$ in $D$ a unique real number denoted by $f(x, y, z)$.*

Examples of functions of 3 variables:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$u = xyz$$

$$T(s_1, s_2, s_3) = \ln s_1 + 12 s_2 - s_3^{-5}.$$

Emphasize that domains of functions of three variables are regions in three dimensional space.

The graph of function of 3 variables is in $\mathbb{R}^4$ $\Rightarrow$ level surfaces are important

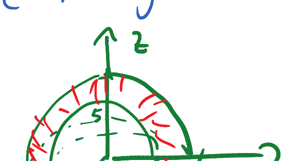


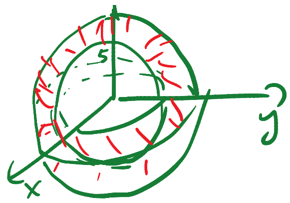
EXAMPLE 9. *Find the domain of the following function:*

$$f(x, y, z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2 + y^2 + z^2 - 25}}.$$

$$\begin{cases} 36 - x^2 - y^2 - z^2 > 0 \Rightarrow x^2 + y^2 + z^2 < 36 \\ x^2 + y^2 + z^2 - 25 > 0 \Rightarrow x^2 + y^2 + z^2 > 25 \end{cases} \Leftrightarrow \underbrace{25 < x^2 + y^2 + z^2 < 36}$$

the region between the sphere of radius 5 and the sphere of radius 6






the sphere of radius 5
and the sphere of radius 6
(not including the boundary spheres)
(Because it is in $\mathbb{R}^4$)

Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

$$f(x, y, z) = k$$

where $k$ is a constant in the range of $f$. If the point $(x, y, z)$ moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 10. *Find the level surfaces of the function* $u = x^2 + y^2 - z$.

The level surface is $x^2+y^2-z=k$ or

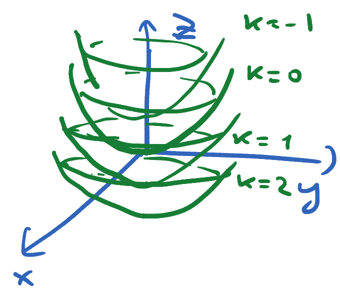
$z = x^2+y^2-k \rightarrow$ circular paraboloids

$k=0$ $\quad z = x^2+y^2$

$k=1$ $\quad z = x^2+y^2-1$

$k=2$ $\quad z = x^2+y^2-2$

$k=-1$ $\quad z = x^2+y^2+1$




REMARK 11. For any function there exist a unique level curve (surface) through given point!!!