F19_LN_14_1

Tuesday, September 17, 2019 8:50 PM

14.1: Functions of Several Variables

Consider the following formulas:

$$z = 2 - x - 4y \tag{1}$$

$$z = x^2 + y^2 \tag{2}$$

$$z = \sqrt{x^2 + y^2} \tag{3}$$

$$z = \sqrt{1 - x^2 - y^2} \tag{4}$$

ordered pair of independent variables (x, y) (input)	$\longrightarrow$	$f(x,y)$	$\longrightarrow$	dependent variable z (output)

subset

DEFINITION 1. *Let $D \subset \mathbb{R}^2$. A* **function** *f* **of two variables** *is a rule that assigns to each ordered pair (x, y) in D a unique real number denoted by $f(x, y)$.*

The set D is the **domain** *of f and the* **range** *of f is the set of values that f takes on, that is $\{f(x, y)|(x, y) \in D\}$.*

REMARK 2. Obviously, one can choose the independent variables arbitrary, for example, $x = f(y, z)$.

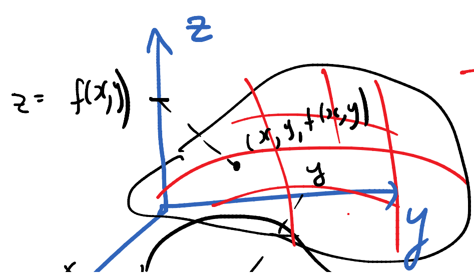
- **GRAPH of** $f(x, y)$.

  Recall that a graph of a function f of one variable is a curve C with equation $y = f(x)$.

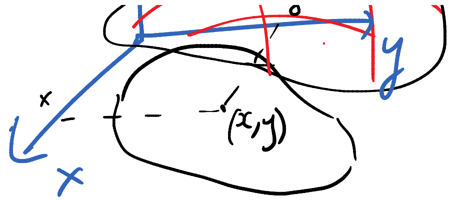
DEFINITION 3. *The* **graph** *of f with domain D is the set:*

$$S = \left\{(x, y, z) \in \mathbb{R}^3 | z = f(x, y). \ \ (x, y) \in D\right\}.$$

The graph of a function f of two variables is a surface S in three dimensional space with equation $z = f(x, y)$.



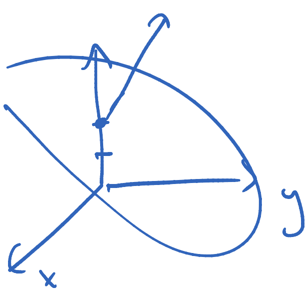
→ the graph is a surface



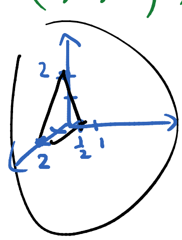
EXAMPLE 4. *Find the domain and sketch the graph of the functions (1)-(4). What is the range?*

(1) $z = 2 - x - 4y$
$D =$ $\mathbb{R}^2$

The graph is a plane ($x+4y+z=2$) orthogonal to $\langle 1,4,1\rangle$ and passing through the point $(0,0,2)$ (I plugged $x=y=0$)

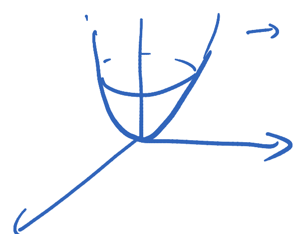


The other way is to find 3 points of the plane (for example, axes intercepts)
$(0,0,2)$, $(0,\frac{1}{2},0)$, $(2,0,0)$



The range is $\mathbb{R}$ (seen from the graph)
Algebraic explanation
For example if $y=0$ and $x=2-z$ then $f(2-z,0)=z$

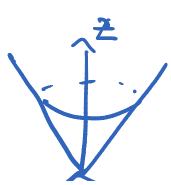
(2) $z = x^2 + y^2$
$D =$ $\mathbb{R}^2$



→ circular paraboloid

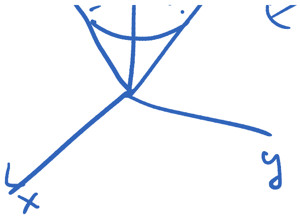
range $= \{z : z \geq 0\}$

(3) $z = \sqrt{x^2+y^2}$ $\Rightarrow z^2 = x^2+y^2$ & $z \geq 0$
$D =$ $\mathbb{R}^2$



range $=$
$= \{z : z \geq 0\}$

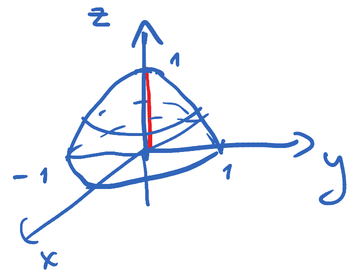
(4) $z = \sqrt{\underbrace{1-x^2-y^2}_{\geq 0}}$ $\overset{\text{square}}{\Longleftrightarrow}$ $\begin{cases} z^2 = 1-x^2-y^2 \\ z \geq 0 \end{cases}$
$D =$ $\{(x,y) : 1-x^2-y^2 \geq 0\}$
$= \{\underbrace{(x,y) : x^2+y^2 \leq 1}\}$
the unit disk around the origin.



$=\{z : z \geq 0\}$

the unit disk around the origin, including the boundary circle

$\begin{cases} x^2+y^2+z^2=1 \\ z \geq 0 \end{cases} \Rightarrow$ the upper hemisphere of radius 1

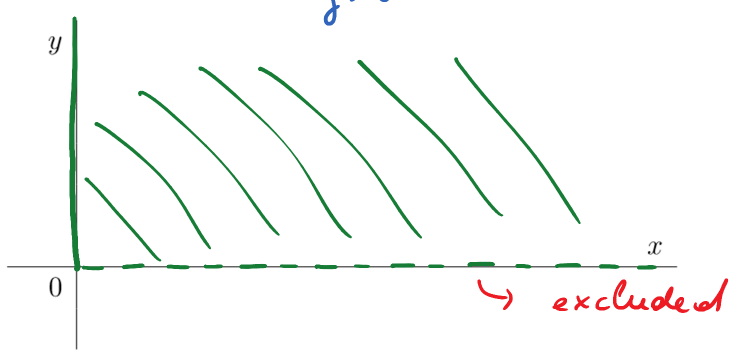


Range $=\{z : 0 \leq z \leq 1\}$

EXAMPLE 5. *Sketch the domain of each of the following:*

(a) $z = \sqrt{x} - \dfrac{5}{\sqrt{y}}$ $\quad x \geq 0$, $\; y > 0$

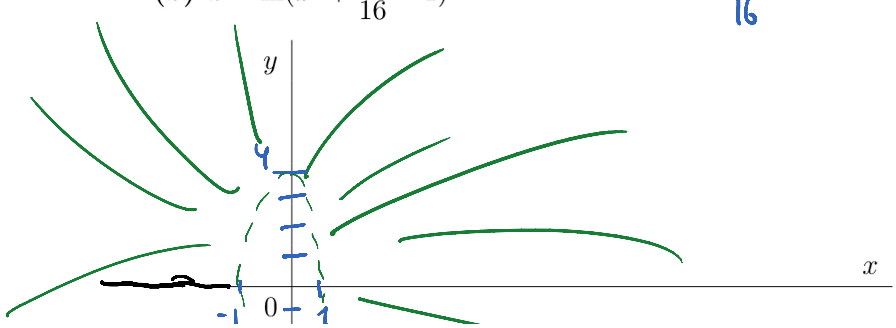
$D = \{(x,y) : x \geq 0, \; y > 0\}$ → the first quadrant without the x-axis



(b) $z = \ln(x^2 + \dfrac{y^2}{16} - 1)$ $\Rightarrow$ $x^2 + \frac{y^2}{16} - 1 > 0 \iff x^2 + \frac{y^2}{16} > 1$

$D = \{(x,y) : x^2 + \frac{y^2}{16} > 1\}$

the region outside of the ellipse $x^2 + \frac{y^2}{16} = 1$, not including the ellipse



$x^2+\frac{y^2}{16}=1$, not including the ellipse

• **LEVEL (CONTOUR) CURVES** method of visualizing functions is the method borrowed from mapmakers. It is a contour map on which points of constant elevation are joined to form level (or contour) curves.

DEFINITION 6. *The* **level (contour) curves** *of a function of two variables are the curves with equations*

$$f(x, y) = k,$$

where k is a constant in the range of f.

A level curve is the locus of all points at which f takes a given value k (it shows where the graph of f has height k).

EXAMPLE 7. *Sketch the level curves of the functions (2) and (3) for the values $k = 0, 1, 2, 3, 4$:*

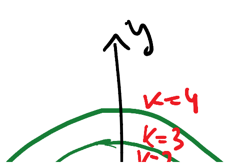
(2) $z = x^2 + y^2$ $\to$ circular paraboloid (3) $z = \sqrt{x^2 + y^2}$ $\to$ upper half a conus

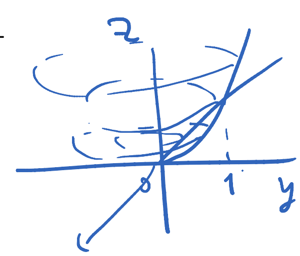
$k \geq 0$ $x^2+y^2=k$
a circle of radius $\sqrt{k}$

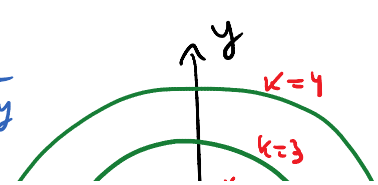
k $\sqrt{x^2+y^2}=k \Rightarrow x^2+y^2=k^2$
a circle of radius k

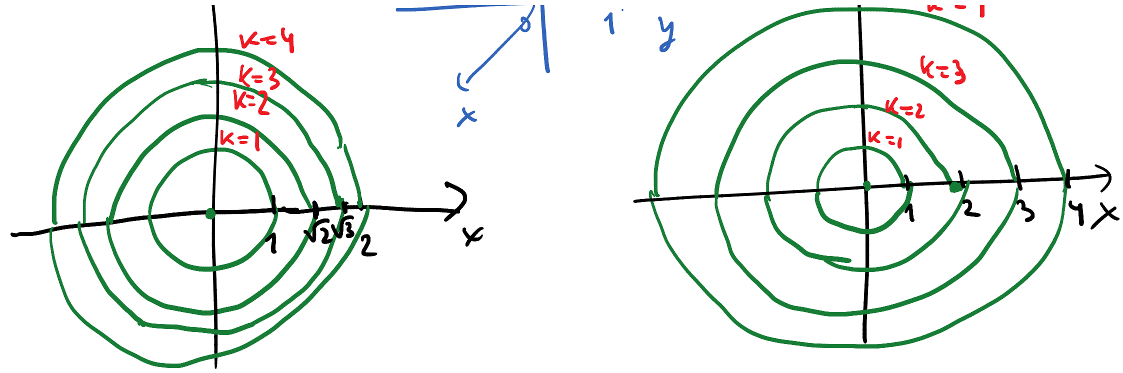
k	The equation	radius
$k=0$	$x^2+y^2=0 \Leftrightarrow (0,0)$	0
$k=1$	$x^2+y^2=1$	1
$k=2$	$x^2+y^2=2$	$\sqrt{2}$
$k=3$	$x^2+y^2=3$	$\sqrt{3}$
$k=4$	$x^2+y^2=4$	2

k	The equation	radius
$k=0$	$(0,0)$	0
$k=1$	$x^2+y^2=1$	1
$k=2$	$x^2+y^2=4$	2
$k=3$	$x^2+y^2=9$	3
$k=4$	$x^2+y^2=16$	4









• **Functions of three variables.**

DEFINITION 8. *Let $D \subset \mathbb{R}^3$. A* **function** *f* **of three variables** *is a rule that assigns to each ordered pair (x, y, z) in D a unique real number denoted by $f(x, y, z)$.*

Examples of functions of 3 variables:

$$f(x, y, z) = x^2 + y^2 + z^2,$$

$$u = xyz$$

$$T(s_1, s_2, s_3) = \ln s_1 + 12s_2 - s_3^{-5}.$$

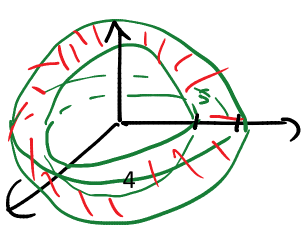
Emphasize that domains of functions of three variables are regions in three dimensional space.

EXAMPLE 9. *Find the domain of the following function:*

$$f(x, y, z) = \frac{\ln(36 - x^2 - y^2 - z^2)}{\sqrt{x^2 + y^2 + z^2 - 25}}.$$

$$\begin{cases} 36 - x^2 - y^2 - z^2 > 0 \Rightarrow x^2+y^2+z^2 < 36 \\ x^2+y^2+z^2 - 25 > 0 \Rightarrow x^2+y^2+z^2 > 25 \end{cases} \iff 25 < x^2+y^2+z^2 < 36$$

the region between spheres of radius 5 & 6 around the origin (not including the spheres)



Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

Note that for functions of three variables it is impossible to visualize its graph. However we can examine them by their **level surfaces**:

because the graph is in $\mathbb{R}^4$

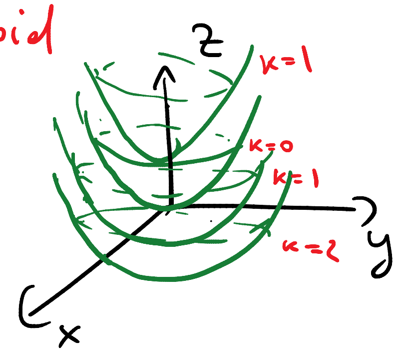
$$f(x, y, z) = k$$

where k is a constant in the range of f. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

EXAMPLE 10. *Find the level surfaces of the function $u = x^2 + y^2 - z$.*

$$x^2 + y^2 - z = K \iff$$

$$z = x^2 + y^2 - K \rightarrow$$ circular paraboloid



$K=0, \quad z = x^2 + y^2$

$K=1, \quad z = x^2 + y^2 - 1$

$K=2, \quad z = x^2 + y^2 - 2$

$K=-1, \quad z = x^2 + y^2 + 1$

REMARK 11. For any function there exist a unique level curve (surface) through given point!!!