F19_LN_14_2

Tuesday, September 17, 2019 8:49 PM

14.2: Limits and Continuity

`Note: A more extensive study of these topics is usually given in advance calculus.`

EXAMPLE 1. *For the function* $f(x,y)=\dfrac{xy}{x^2+y^2}$ *find the limits at* $(0,0)$ *along* $D=\mathbb{R}^2\setminus\{(0,0)\}$

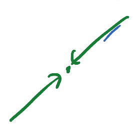
(a) *the x-axis*

along x-axis, i.e. $y=0$: $f(x,0)=\dfrac{x\cdot 0}{x^2+0}=0 \Rightarrow \lim\limits_{x\to 0} f(x,0)=0$

(b) *the y-axis*

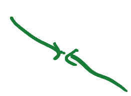
along y-axis i.e. $x=0$: $f(0,y)=\dfrac{0\cdot y}{0+y^2}=0 \Rightarrow \lim\limits_{y\to 0} f(0,y)=0$

(c) *the line* $y=x$



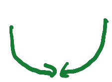
$$f(x,x)=\frac{x^2}{x^2+x^2}=\frac{\cancel{x^2}}{2\cancel{x^2}}=\frac{1}{2}\Rightarrow \lim_{x\to 0} f(x,x)=\frac{1}{2}$$

(d) *the line* $y=-x$



$$f(x,-x)=\frac{x\cdot(-x)}{x^2+(-x)^2}=-\frac{x^2}{2x^2}=-\frac{1}{2}\Rightarrow \lim_{x\to 0} f(x,-x)=-\frac{1}{2}$$

(e) *the parabola* $y=x^2$



$$f(x,x^2)=\frac{x\cdot x^2}{x^2+x^4}=\frac{x^{\cancel{3}}}{\cancel{x^2}(1+x^2)}=\frac{x}{1+x^2}\Rightarrow \lim_{x\to 0} f(x,x^2)=0$$

a) & c) $\Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist

The statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

is intended to convey the idea that the value of the function $f(x,y)$ can be made as close as we like to the number L by restricting the point (x,y) to be sufficiently close to (but different from) the point (x_0,y_0).

We note without proof that the standard properties of limits hold for limits along curves and for general limits of functions of two variables, so that computations involving such limits can be performed in usual way.

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EXAMPLE 2. *Find*

$$\lim_{(x,y)\to(1,3)} (4x^3y - 2018) = \underbrace{4 \cdot 1^3 \cdot 3}_{12} - 2018 = -2006$$

THEOREM 3.

- *If* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ *then* $f(x,y) \to L$ *as* $(x,y) \to (x_0,y_0)$ *along any smooth curve.*
- *If the limit of* $f(x,y)$ *fails to exist as* $(x,y) \to (x_0,y_0)$ *along some smooth curve, or if* $f(x,y)$ *has different limits as* $(x,y) \to (x_0,y_0)$ *along two different smooth curves, then the limit of* $f(x,y)$ *does not exits as* $(x,y) \to (x_0,y_0)$.

Another example $\frac{x^3y}{x^6+y^4}$

homogeneous all terms are of the same degree

EXAMPLE 4. *For the function* $f(x,y) = \dfrac{xy}{x^2+y^2}$ *discuss its limit as* $(x,y) \to (0,0)$.

See analysis in example 1 $\longrightarrow$ the limit does not exist

In general $\dfrac{\text{hom. polynomial of degree } k}{\text{hom. polynomial of degree } k} \xrightarrow[(x,y)\to 0]{}$ does not exists (and numerator is not a constant $\times$ denominator)

DEFINITION 5. *A function* $f(x,y)$ *is continuous at the point* (x_0,y_0) *if*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Roughly speaking, a function will be continuous at a point if the graph does not have any holes or breaks at that point.

All the standard functions that we know to be continuous are still continuous even if we are plugging in more than one variable now. We just need to watch out for division by zero, square roots of negative numbers, logarithms of zero or negative numbers, etc.

Recognizing Continuous functions:

- A polynomial of function of (x,y) is continuous.
- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient ~~of~~ of continuous functions is continuous, except where the denominator is zero.

EXAMPLE 6. *Confirm that the following functions are all continuous everywhere:*

$$f(x,y) = ye^{x+y} + \sqrt[3]{x} \qquad g(x,y) = \frac{x^3y+6}{\ldots} \qquad h(x,y) = |1-xy| + \sin(x^5y^4)$$

EXAMPLE 6. *Confirm that the following functions are all continuous everywhere:*

$$f(x,y) = ye^{x+y} + \sqrt[3]{x}, \quad g(x,y) = \frac{x^3y+6}{1+x^2+y^2}, \quad h(x,y) = |1-xy| + \sin(x^5y^4)$$

because it is obtained by composition, product and sum of continuous functions

(also quotient by a non-zero function)

EXAMPLE 7. *Evaluate* $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{xy}{x^2+y^2} \underset{\substack{\text{arithmetic}\\ \text{of limits}}}{=} \frac{-1\cdot 1}{(-1)^2+1^2} = -\frac{1}{2}$

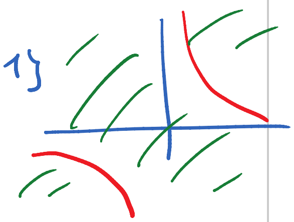
EXAMPLE 8. *Find all points where the function* $f(x,y) = \dfrac{xy+12y^4}{xy-1}$ *is continuous.*

The problematic points are where $xy-1=0 \Leftrightarrow xy=1 \Leftrightarrow y=\frac{1}{x}$

(note that at this point the numerator $= 1+12y^4>0$) $\Rightarrow$

at these points function is not continuous, because $\lim \to \infty$, but in all other points by the arithmetic of limits it is continuous:

$\{(x,y) : xy \neq 1\}$



EXAMPLE 9. *Evaluate* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$ $\to$ deg 3, $\to$ deg 2

Use polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{x^2y}{x^2+y^2} = \frac{r^3\cos^2\theta\sin\theta}{\cancel{r^2}\underbrace{(\cos^2\theta+\sin^2\theta)}_{1}} = r\underbrace{\cos^2\theta}_{\leq 1}\sin\theta \quad |\sin\theta|\leq 1$$

$$\left|\frac{x^2y}{x^2+y^2}\right| = r\underbrace{|\cos^2\theta|}_{\leq 1}\underbrace{|\sin\theta|}_{\leq 1} \leq r = \sqrt{x^2+y^2}$$

$$-\sqrt{x^2+y^2} \leq \frac{x^2y}{x^2+y^2} \leq \sqrt{x^2+y^2}, \quad \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$$

$$-\sqrt{x^2+y^2} < \frac{x^2 y}{x^2+y^2} \le \sqrt{x^2+y^2}, \quad \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

$(x,y)\to(0,0)$

0

Similarly $\dfrac{\sin x \cdot xy}{x^2+y^2}$

$$-\sin \le \frac{\sin x \cdot xy}{x^2+y^2} \le \sin x$$

→ bounded

0

$$\underbrace{f(x,y)}_{\substack{\downarrow \\ 0 \text{ as } (x,y)\to(0,0)}} \cdot \underbrace{g(x,y)}_{\text{bounded}} \xrightarrow[(x,y)\to(0,0)]{} 0$$