F19_LN_14_2

Tuesday, September 17, 2019 8:51 PM

14.2: Limits and Continuity

Note: A more extensive study of these topics is usually given in advance calculus.

EXAMPLE 1. *For the function* $f(x,y) = \dfrac{xy}{x^2+y^2}$ *find the limits at* $(0,0)$ *along*

$\to$ the domain is $\mathbb{R}^2 \setminus \{(0,0)\}$

(a) *the* x*-axis*

$y=0$

$f(x,0) = 0 \Rightarrow \lim_{x\to 0} f(x,0) = 0$

(b) *the* y*-axis*

$x=0$

$f(0,y) = 0 \Rightarrow \lim_{y\to 0} f(0,y) = 0$

(c) *the line* $y = x$

$f(x,x) = \dfrac{x^2}{x^2+x^2} = \dfrac{\cancel{x^2}}{2\cancel{x^2}} = \dfrac{1}{2} \Rightarrow \lim_{x\to 0} f(x,x) = \dfrac{1}{2}$

(a) & (c) $\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exists

$\frac{1}{2} \neq 0$

(d) *the line* $y = -x$

$f(x,-x) = \dfrac{x\cdot(-x)}{x^2+(-x)^2} = -\dfrac{x^2}{2x^2} = -\dfrac{1}{2} \Rightarrow \lim_{x\to 0} f(x,-x) = -\dfrac{1}{2}$

(e) *the parabola* $y = x^2$

$f(x,x^2) = \dfrac{x\cdot x^2}{x^2+x^4} = \dfrac{x^{\cancel{3}}}{\cancel{x^2}(1+x^2)} = \dfrac{x}{1+x^2} \Rightarrow \lim_{x\to 0} f(x,x^2) = 0$

The statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

is intended to convey the idea that the value of the function $f(x,y)$ can be made as close as we like to the number L by restricting the point (x,y) to be sufficiently close to (but different from) the point (x_0,y_0).

We note without proof that the standard properties of limits hold for limits along curves and for general limits of functions of two variables, so that computations involving such limits can be performed in usual way.

for general limits of functions of two variables, so that computations involving such limits can be performed in usual way.

EXAMPLE 2. *Find*

$$\lim_{(x,y)\to(1,3)} (4x^3y - 2018) = 4\cdot 1^3\cdot 3 - 2018 = -2006$$

arithmetic of limits

THEOREM 3.

- *If* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ *then* $f(x,y)\to L$ *as* $(x,y)\to(x_0,y_0)$ *along any smooth curve.*
- *If the limit of* $f(x,y)$ *fails to exist as* $(x,y)\to(x_0,y_0)$ *along some smooth curve, or if* $f(x,y)$ *has different limits as* $(x,y)\to(x_0,y_0)$ *along two different smooth curves, then the limit of* $f(x,y)$ *does not exits as* $(x,y)\to(x_0,y_0)$.

similar example $\frac{x^3y}{x^4+y^4}$ $\to$ deg 4, $\to$ deg 4

EXAMPLE 4. *For the function* $f(x,y) = \dfrac{xy}{x^2+y^2}$ *discuss its limit as* $(x,y)\to(0,0)$.

See analysis in example : the limit does not exist as $\lim_{x\to 0} f(x,0) = 0 \neq \frac{1}{2} = \lim_{x\to 0} f(x,x)$

DEFINITION 5. *A function* $f(x,y)$ *is continuous at the point* (x_0,y_0) *if*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Roughly speaking, a function will be continuous at a point if the graph does not have any holes or breaks at that point.

All the standard functions that we know to be continuous are still continuous even if we are plugging in more than one variable now. We just need to watch out for division by zero, square roots of negative numbers, logarithms of zero or negative numbers, etc.

Recognizing Continuous functions:

- A polynomial of function of (x,y) is continuous.
- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of of continuous functions is continuous, except where the denominator is zero.

EXAMPLE 6. *Confirm that the following functions are all continuous everywhere:*

$$f(x,y) = ye^{x+y} + \sqrt[3]{x} \qquad g(x,y) = \frac{x^3y+6}{\ldots} \qquad h(x,y) = |1-xy| + \sin(x^5y^4)$$

EXAMPLE 6. *Confirm that the following functions are all continuous everywhere:*

$$f(x,y) = ye^{x+y} + \sqrt[3]{x}, \quad g(x,y) = \frac{x^3y+6}{1+x^2+y^2} , \quad \text{where } 1+x^2+y^2 > 1, \quad h(x,y) = |1-xy| + \sin(x^5y^4)$$

all these functions are continuous according to above rules

EXAMPLE 7. *Evaluate* $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{xy}{x^2+y^2} = \frac{-1\cdot 1}{(-1)^2+1^2} = -\frac{1}{2}$

arithmetic of limits

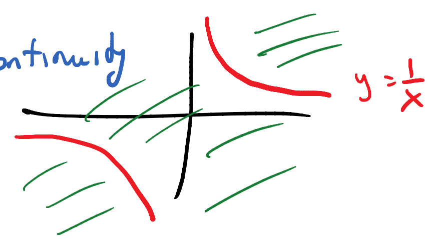
EXAMPLE 8. *Find all points where the function* $f(x,y) = \dfrac{xy+12y^4}{xy-1}$ *is continuous.*

The problem might be when $xy-1=0 \iff xy=1 \iff y=\frac{1}{x}$

At these point the numerator is $1+12y^4>0$

$\Rightarrow$ approaching these point we get $+\infty \Rightarrow$ no continuity

If $xy\neq 1$ then f is continuous by the arithmetics of limits $\Rightarrow$ Answer: $\{(x,y): xy\neq 1\}$



EXAMPLE 9. *Evaluate* $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$ $\begin{matrix}\to \text{deg } 3\\ \to \text{deg } 2\end{matrix}$

Use polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$$\frac{x^2y}{x^2+y^2} = \frac{r^{\cancel{3}}\cos^2\theta\sin\theta}{\cancel{r^2}(\underbrace{\cos^2\theta+\sin^2\theta}_{1})} = r\cos^2\theta\sin\theta$$

$$\left|\frac{x^2y}{x^2+y^2}\right| = r\underbrace{\cos^2\theta}_{\leq 1}\underbrace{|\sin\theta|}_{\leq 1} \leq r = \sqrt{x^2+y^2} \Rightarrow$$

$$-\sqrt{x^2+y^2} \leq \frac{x^2y}{x^2+y^2} \leq \sqrt{x^2+y^2} \Rightarrow$$

$(x,y)\to(0,0)$

$$x^2+y^2$$

$(x,y)\to(0,0)$ $(x,y)\to(0,0)$

Sandwich $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$

Similarly if $\dfrac{\sin x \cdot xy}{x^2+y^2}$ $\qquad \left| \dfrac{\sin x \; xy}{x^2+y^2} \right| \le |\sin x| \to 0$

In general, if $f(x,y) \xrightarrow[(x,y)\to(x_0,y_0)]{} 0$ & $g(x,y)$ is bounded in the punctered neighb of (x_0, y_0) (i.e $|g(x,y)| \le M$ there) then

$$f(x,y)\, g(x,y) \xrightarrow[(x,y)\to(x_0,y_0)]{} 0$$