F19_LN_14_3

Sunday, September 22, 2019 10:16 PM



## 14.3: Partial Derivatives

DEFINITION 1. *If $f$ is a function of two variables, its* **partial derivatives** *are the functions $f_x$ and $f_y$ defined by*

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

*Conclusion:* $f_x(x,y)$ represents the *rate of change* of the function $f(x,y)$ as we change $x$ and hold $y$ fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change $y$ and hold $x$ fixed.

**Notations for partial derivatives:** If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

*RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$:*

1. To find $f_x$, regard $y$ as a constant and differentiate $f(x, y)$ with respect to $x$.
2. To find $f_y$, regard $x$ as a constant and differentiate $f(x, y)$ with respect to $y$.

EXAMPLE 2. *If $f(x, y) = x^3 + y^5 e^x$ find $f_x(0, 1)$ and $f_y(0, 1)$.*

$$f_x = 3x^2 + y^5 e^x \Rightarrow f_x(0,1) = 3 \cdot 0 + 1 \cdot e^0 = 1$$

$$f_y = 5y^4 e^x \Rightarrow f_y(0,1) = 5 \cdot 1 \cdot e^0 = 5$$

EXAMPLE 3. *Find all of the first order partial derivatives for the following functions:*

**(a)** $z(x, y) = x^3 \sin(xy)$

$$z_x = \underbrace{\frac{\partial}{\partial x}(x^3)}_{3x^2} \sin(xy) + x^3 \underbrace{\frac{\partial}{\partial x} \sin(xy)}_{y \cos xy} = 3x^2 \sin(xy) + x^3 y \cos(xy)$$

$$\underbrace{\phantom{xx}}_{3x^2} \qquad \underbrace{\phantom{xx}}_{y\cos xy}$$

$$z_y = x^3 \underbrace{\frac{\partial}{\partial y}\sin(xy)}_{x\cos(xy)} = x^4\cos(xy)$$

**(c)** $u(x, y, z) = ye^{xyz}$

$$\frac{\partial}{\partial x} u(x,y,z) \underset{\substack{y\,\&\,z \\ \text{are constant}}}{=} y \underbrace{\frac{\partial}{\partial x} e^{xyz}}_{yz\,e^{xyz}} = y^2 z e^{xyz}$$

$$\frac{\partial}{\partial y} u \underset{\substack{x\,\&\,z \\ \text{are constant}}}{=} \underbrace{\frac{\partial}{\partial y}(y)}_{1} \cdot e^{xyz} + y \cdot \underbrace{\frac{\partial}{\partial y} e^{xyz}}_{xz\,e^{xyz}} = e^{xyz} + xyz\, e^{xyz} = (1+xyz)e^{xyz}$$

$$\frac{\partial}{\partial z} u \underset{x\,\&\,y \text{ are constant}}{=} y \frac{\partial}{\partial z} e^{xyz} = y \cdot xy\, e^{xyz} = xy^2 e^{xyz}$$



EXAMPLE 4. *The temperature at a point* $(x, y)$ *on a flat metal plate is given by*

$$T(x,y) = \frac{80}{1+x^2+y^2}, = 80\,(\underbrace{1+x^2+y^2}_{z})^{-1}$$

*where* $T$ *is measured in* °C *and* $x, y$ *in meters. Find the rate of change of temperature with respect to distance at the point* $(1, 2)$ *in the* $y$*-direction.*

$$\frac{\partial T}{\partial y}(1,2)\ ?$$

chain rule

$$\frac{\partial T}{\partial y} \underset{x \text{ is constant}}{=} 80\cdot(-1)\,(1+x^2+y^2)^{-2} \cdot \underbrace{2y}_{\frac{\partial z}{\partial y}} = -\frac{160y}{(1+x^2+y^2)^2} \Rightarrow$$

$$\frac{d}{dz}(z^{-1}) = -z^{-2}$$

Plug $(x,y) = (1,2)$

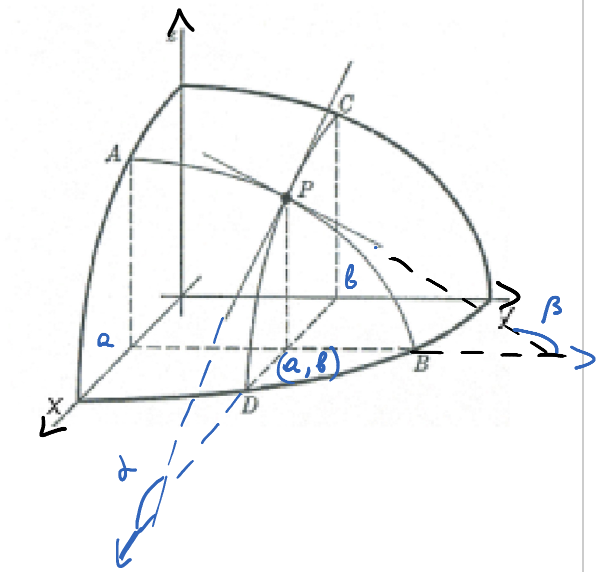
$$\frac{\partial T}{\partial y}(1,2) = -\frac{160\cdot 2}{(1+1+4)^2} = -\frac{\cancelto{80}{320}}{\cancelto{9}{36}} = -\frac{80}{9}\ {}^\circ\mathrm{C}/\mathrm{m}$$

**Geometric interpretation of partial derivatives:** Partial derivatives are the *slopes of traces*:

- $f_x(a, b)$ is the slope of the trace of the traces.

- $f_x(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $y = b$ at the point $(a,b)$.

$f_x(a,b) = \tan\alpha$ (slope of the tangent line to the $y=b$ trace of the graph




- $f_y(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $x = a$ at $(a,b)$.

$f_y(a,b) = \tan\beta$



EXAMPLE 5. *If* $f(x,y) = \sqrt{\underbrace{4 - x^2 - 4y^2}_{z}} = (4-x^2-4y^2)^{1/2}$, *find* $f_x(1,0)$ *and* $f_y(1,0)$ *and interpret these numbers as slopes. Illustrate with sketches.*

$$f_x = \underbrace{\tfrac{1}{2}(4-x^2-4y^2)^{-1/2}}_{\frac{d}{dz} z^{1/2} = \frac{1}{2} z^{-1/2}} \cdot \underbrace{(-\cancel{2}x)}_{\frac{\partial z}{\partial x}} = -\frac{x}{\sqrt{4-x^2-4y^2}}$$

chain rule

$z = \sqrt{4-x^2-4y^2}$

$z^2 = 4 - x^2 - 4y^2,\ z \geq 0$

$x^2 + 4y^2 + z^2 = 4$

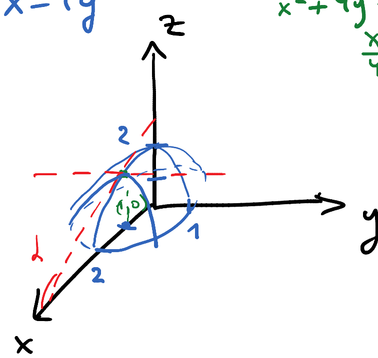
$\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1$

$z \geq 0$

$\hookrightarrow$ top half of the ellipsoid

$$f_x(1,0) = -\frac{1}{\sqrt{4-1}} = -\frac{1}{\sqrt{3}} = \tan\alpha$$

$$\Rightarrow \alpha = \frac{5\pi}{6}$$




$$f_y = \frac{1}{\cancel{2}}(4-x^2-4y^2)^{-1/2} \cdot -\overset{4}{\cancel{8}}y = -\frac{4y}{\sqrt{4-x^2-4y^2}}$$

$f_y(1,0) = 0 \Rightarrow$ the tangen line at $(1,0)$ to the trace of the graph of $f$ when $x=1$ is

$f_y(1,0) = 0 \Rightarrow$ the tangen... the graph of $f$ when $x=1$ is horizontal

**Higher derivatives:** Since both of the first order partial derivatives for $f(x,y)$ are also functions of $x$ and $y$, so we can in turn differentiate each with respect to $x$ or $y$. We use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

mixed derivative

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$



EXAMPLE 6. *Find the second partial derivatives of*

$$f(x,y) = y^3 + 5y^2e^{4x} - \cos(x^2).$$

$$f_x = 0 + 5y^2 \cdot 4e^{4x} - (-\sin(x^2)) \cdot 2x = 20y^2e^{4x} + 2x\sin(x^2)$$

$$f_y = 3y^2 + 10ye^{4x}$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(20y^2e^{4x} + 2x\sin(x^2)\right) = 80y^2e^{4x} +$$

$$+ 2\sin(x^2) + 2x \cdot 2x \cdot \cos(x^2) = \boxed{80y^2e^{4x} + 2\sin(x^2) + 4x^2\cos(x^2)}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(3y^2 + 10ye^{4x}\right) = \boxed{6y + 10e^{4x}}$$

$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(3y^2 + 10 y e^{4x}) = \boxed{6y + 10 e^{4x}}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\Big( 20y^2 e^{4x} + \underbrace{2x\sin(x^2)}_{\text{Indepent of } y} \Big) =$

$= \boxed{40y\, e^{4x}}$ =

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\big(\underbrace{3y^2}_{\text{Indep. of } x} + 10 y e^{4x}\big) = \boxed{40 y e^{4x}}$

$f_{xy} = f_{yx}$ and this is not a coincident

**Clairaut's Theorem.** *Suppose $f$ is defined on a disk $D$ that contains the point $(a, b)$. If the functions $f_{xy}$ and $f_{yx}$ are both continuous on $D$ then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Partial derivative of order three or higher can also be defined. For instance,

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2}.$$

Using Clairaut's Theorem one can show that if the functions $f_{yyx}$, $f_{xyy}$ and $f_{yxy}$ are continuous then

$f_{yyx} = f_{xyy} = f_{yxy}$ (if these derivatives are continuous

$\neq$

$f_{yxx}$ (in general)



EXAMPLE 7. *Find the indicated derivative for*

$$f(x, y, z) = \cos(xy + z).$$

**(a)** $f_{xy}$

chain rule

$f_x = -\sin(xy+z)\cdot y = -y\sin(xy+z)$

$f_{xy} = \frac{\partial}{\partial y}\big(-y\sin(xy+z)\big) = -\sin(xy+z) - y\underbrace{\frac{\partial}{\partial y}\sin(xy+z)}_{x\cos(xy+z)} =$

product rule

$= -\sin(xy+z) - xy\cos(xy+z)$

$x\cos(xy+z)$

product rule

$= -\sin(xy+z) - xy\cos(xy+z)$

(b) $f_{zxy} = f_{xyz} = \frac{\partial}{\partial z}\left(-\sin(xy+z) - xy\cos(xy+z)\right) =$

Clairaut's Thm $= -\cos(xy+z) + xy\sin(xy+z)$

EXAMPLE 8. *If $f$ and $g$ are twice differentiable functions of a single variable, show that the function*

$$u(x,t) = f(x+at) + g(x-at)$$

*is a solution of the wave equation*

$$u_{tt} = a^2 u_{xx}.$$ → wave equation

$u_t = f'(x+at)a + g'(x-at)\cdot(-a)$ → Try to do it by yourself.