F19_LN_14_3

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14.3: Partial Derivatives

DEFINITION 1. *If f is a function of two variables, its* **partial derivatives** *are the functions f_x and f_y defined by*

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Conclusion: $f_x(x,y)$ represents the *rate of change* of the function $f(x,y)$ as we change x and hold y fixed while $f_y(x,y)$ represents the rate of change of $f(x,y)$ as we change y and hold x fixed.

Notations for partial derivatives: If $z = f(x, y)$, we write

$$f_x(x,y) = f_x = f'_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

RULE FOR FINDING PARTIAL DERIVATIVES OF $z = f(x, y)$:

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE 2. *If $f(x, y) = x^3 + y^5 e^x$ find $f_x(0, 1)$ and $f_y(0, 1)$.*

$$f_x(x,y) \underset{\substack{\downarrow \\ y \text{ is a constant}}}{=} 3x^2 + y^5 e^x \Rightarrow f_x(0,1) = 3\cdot 0 + 1e^0 = 1$$

$$f_y(x,y) \underset{\substack{\downarrow \\ x \text{ is a constant}}}{=} 0 + 5y^4 e^x = 5y^4 e^x \Rightarrow f_y(0,1) = 5\cdot 1\cdot e^0 = 5$$

EXAMPLE 3. *Find all of the first order partial derivatives for the following functions:*

(a) $z(x, y) = x^3 \sin(xy)$

$$z_x \overset{\text{product rule}}{\underset{\substack{\downarrow \\ y \text{ is constant}}}{=}} \underbrace{\frac{\partial}{\partial x}(x^3)}_{3x^2} \cdot \sin xy + x^3 \cdot \underbrace{\frac{\partial}{\partial x} \sin(xy)}_{y \cos xy} = \boxed{3x^2 \sin xy + x^3 y \cos xy}$$

$$z_x \underset{\substack{\downarrow \\ y \text{ is constant}}}{=} \underbrace{\frac{\partial}{\partial x}(x^3)}_{3x^2} \cdot \sin xy + x^3 \cdot \underbrace{\frac{\partial}{\partial x} \sin(xy)}_{y\cos xy} = \boxed{3x^2 \sin xy + x^3 y \cos xy}$$

$$z_y \underset{\substack{\downarrow \\ x \text{ is constant}}}{=} x^3 \underbrace{\frac{\partial}{\partial y} \sin(xy)}_{x\cos(xy)} = \boxed{x^4 \cos xy}$$

(c) $u(x, y, z) = ye^{xyz}$

$$\frac{\partial u}{\partial x} \underset{\substack{\downarrow \\ y \,\&\, z \text{ are constant}}}{=} y \underbrace{\frac{\partial}{\partial x} e^{xyz}}_{yz e^{xyz}} = y^2 z e^{xyz},$$

$$\frac{\partial u}{\partial y} \underset{\substack{\downarrow \\ x \,\&\, z \text{ are constant}}}{\overset{\text{product rule}}{=}} \underbrace{\frac{\partial}{\partial y}(y)}_{1} \cdot e^{xyz} + y \cdot \underbrace{\frac{\partial}{\partial y} e^{xyz}}_{xz e^{xyz}} = \boxed{e^{xyz} + xyz e^{xyz} = (1+xyz)e^{xyz}}$$

$$\frac{\partial u}{\partial z} \underset{x \,\&\, y \text{ are constant}}{=} y \underbrace{\frac{\partial}{\partial z} e^{xyz}}_{xy e^{xyz}} = xy^2 e^{xyz}$$

EXAMPLE 4. *The temperature at a point* (x, y) *on a flat metal plate is given by*

$$T(x, y) = \frac{80}{1 + x^2 + y^2}, = 80 \cdot (\underbrace{1 + x^2 + y^2}_{u})^{-1}$$

where T *is measured in* °C *and* x, y *in meters. Find the rate of change of temperature with respect to distance at the point* $(1, 2)$ *in the* y*-direction.*

$$\frac{\partial T}{\partial y} \underset{\substack{\downarrow \\ \text{chain rule}}}{=} \underbrace{-80(1+x^2+y^2)^{-2}}_{\frac{\partial}{\partial u}(80u^{-1})} \cdot \underbrace{\frac{\partial}{\partial y}(1+x^2+y^2)}_{\frac{\partial}{\partial y} u = 2y} = -\frac{80}{(1+x^2+y^2)^2} \cdot 2y = -\frac{160y}{(1+x^2+y^2)^2}$$

$$\xrightarrow{\text{plug } (x,y)=(1,2)} \frac{\partial T}{\partial y}(1,2) = -\frac{160 \cdot 2}{(\underbrace{1+1+4}_{36\ 18})^2} = -\frac{80}{9}\ {}^\circ\mathrm{C}/\mathrm{m}$$

Geometric interpretation of partial derivatives: Partial derivatives are the *slopes of traces*:

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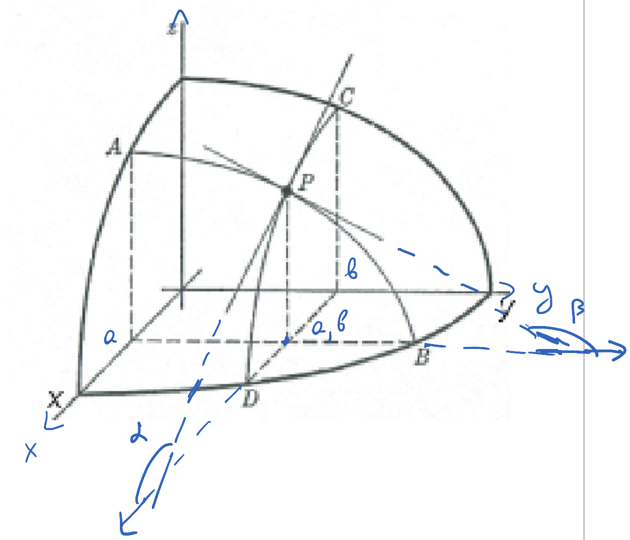
- $f_x(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $y = b$ at the point (a,b).

$f_x(a,b) = \tan\alpha$ (the slope)



- $f_y(a,b)$ is the slope of the trace of the graph of $z = f(x,y)$ for the plane $x = a$ at (a,b).

$f_y(a,b) = \tan\beta$

EXAMPLE 5. *If* $f(x,y) = \sqrt{\underbrace{4 - x^2 - 4y^2}_{z}} = z^{1/2}$, *find* $f_x(1,0)$ *and* $f_y(1,0)$ *and interpret these numbers as slopes. Illustrate with sketches.*

$f_x = \underbrace{\frac{1}{\cancel{2}}(4-x^2-4y^2)^{-1/2}}_{\frac{d}{dz} z^{1/2} = \frac{1}{2} z^{-1/2}} \cdot \underbrace{(-\cancel{2}x)}_{\frac{\partial z}{\partial x}} = -\frac{x}{\sqrt{4-x^2-4y^2}}$

chain rule

$z = \sqrt{4-x^2-4y^2}$

$\Downarrow$ square it

$z^2 = 4 - x^2 - 4y^2,\ z \geq 0$

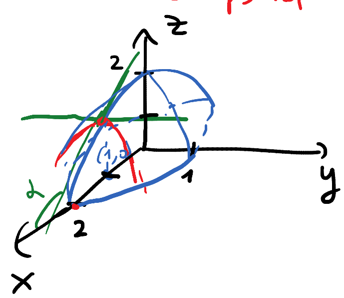
$x^2 + 4y^2 + z^2 = 4,\ z \geq 0$

$\underbrace{\frac{x^2}{4} + y^2 + \frac{z^2}{4} = 1,\ z \geq 0}$

top half of an ellipsoid

$f_x(1,0) = -\frac{1}{\sqrt{4-1}} = -\frac{1}{\sqrt{3}} = \tan\alpha \Rightarrow$

$\alpha = \frac{5\pi}{6}$



$f_y = \frac{1}{\cancel{2}}(4-x^2-4y^2)^{-1/2} \cdot (-\cancel{8}^{4}y) = -\frac{4y}{\sqrt{4-x^2-4y^2}}$

$f_y(1,0) = 0 \Rightarrow$ the tangent li...

$f_y(1,0) = 0 \Rightarrow$ the tangent line to the graph of $f(1,y)$ is horizontal

Higher derivatives: Since both of the first order partial derivatives for $f(x,y)$ are also functions of x and y, so we can in turn differentiate each with respect to x or y. We use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

mixed derivatives

$$\left\{\begin{aligned} (f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\ (f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \end{aligned}\right.$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

EXAMPLE 6. *Find the second partial derivatives of*

$$f(x,y) = y^3 + 5y^2e^{4x} - \cos(x^2).$$

1st order derivatives

$$\left\{\begin{aligned} \frac{\partial}{\partial x} f &= 0 + 5y^2 \cdot 4e^{4x} - (-\sin(x^2)) \cdot 2x = 20y^2e^{4x} + 2x\sin(x^2) \\ \frac{\partial f}{\partial y} &= 3y^2 + 10ye^{4x} \end{aligned}\right.$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(20y^2e^{4x} + 2x\sin(x^2)\right) = \boxed{80y^2e^{4x} + 2\sin(x^2) + 4x^2\cos(x^2)}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(3y^2 + 10ye^{4x}\right) = \boxed{6y + 10e^{4x}}$$

$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}(3y^2 + 10ye^{4x}) = \boxed{6y + 10e^{4x}}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(20y^2e^{4x} + \underbrace{2x\sin(x^2)}_{\text{indep. of } y}) = 40ye^{4x}$

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(\underbrace{3y^2}_{\text{Indep of. } x} + 10ye^{4x}) = 40ye^{4x}$

By Clairaut Thm it is superfluous to calculate f_{yx}

$f_{xy} = f_{yx}$ and this is not a coincidence

Clairaut's Theorem. *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Partial derivative of order three or higher can also be defined. For instance,

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{\partial^3 z}{\partial x \partial y^2}.$$

Using Clairaut's Theorem one can show that if the functions f_{yyx}, f_{xyy} and f_{yxy} are continuous then

$f_{yyx} = f_{yxy} = f_{xyy}$

$\neq$ in general f_{xxy}

EXAMPLE 7. *Find the indicated derivative for*

$$f(x, y, z) = \cos(xy + z).$$

(a) f_{xy}

$f_x \overset{\text{Chain rule}}{=} -\sin(xy+z) \cdot y = -y\sin(xy+z)$

$f_{xy} = \frac{\partial}{\partial y}(-y\sin(xy+z)) \underset{\text{Product}}{=} -\sin(xy+z) - y\frac{\partial}{\partial y}\sin(xy+z) =$

$f_{xy} = \frac{\partial}{\partial y}\left(-y\sin(xy+z)\right) = -\sin(xy+z) - y\frac{\partial}{\partial y}\sin(xy+z) =$

Product rule

$$= \underbrace{-\sin(xy+z) - xy\cos(xy+z)}$$

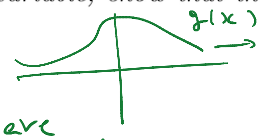
(b) $f_{zxy} = f_{xyz} = \frac{\partial}{\partial z}(f_{xy}) = \frac{\partial}{\partial z}\left(-\sin(xy+z) - xy\cos(xy+z)\right)$

by Clairaut's thm

$$= \boxed{-\cos(xy+z) + xy\sin(xy+z)}$$

EXAMPLE 8. *If f and g are twice differentiable functions of a single variable, show that the function*

$$u(x,t) = f(x+at) + g(x-at)$$



travelling wave

travelling wave

is a solution of the wave equation

$$u_{tt} = a^2 u_{xx}.$$

→ wave equation

$u_t = f'(x+at)\cdot a + g'(x-at)\cdot(-a)$

chain rule

$u_{tt} = f''(x+at)\,a^2 + g''(x-at)\cdot\underbrace{(-a)^2}_{a^2} = a^2\left(f''(x+at) + g''(x-at)\right)$

$\Downarrow$

Similarly $u_{xx} = f''(x+at) + g''(x-at)$

$\Rightarrow \quad u_{tt} = a^2 u_{xx}$