F19_LN_14_4

Thursday, September 26, 2019 10:30 PM

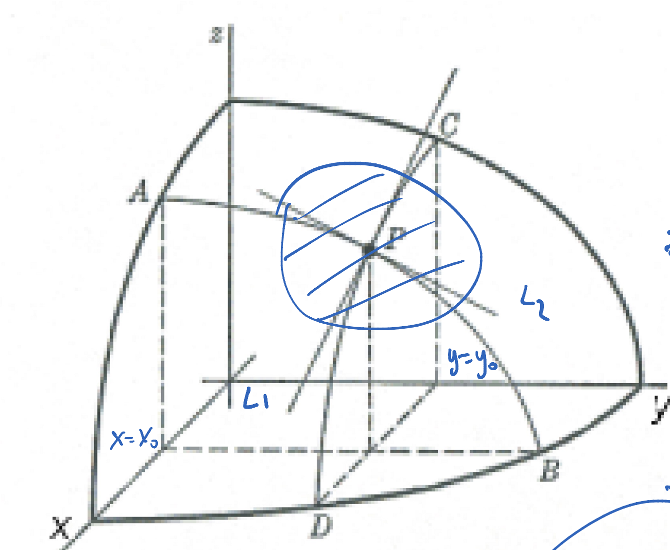
F19_LN_1...

14.4: Tangent Planes and Differentials

Suppose that $f(x, y)$ has continuous first partial derivatives and a surface S has equation $z = f(x, y)$. Let $P(x_0, y_0, z_0)$ be a point on S, i.e. $z_0 = f(x_0, y_0)$.

Denote by C_1 the trace to $f(x, y)$ for the plane $y = y_0$ and denote by C_2 the trace to $f(x, y)$ for the plane $x = x_0$. let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The **tangent plane** to the surface S (or to the graph of $f(x, y)$) at the point P is defined to be the plane that contains both the tangent lines L_1 and L_2.



$z = f(x,y)$

L_1: $y = y_0$

$z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0)$

L_2: $x = x_0$

$z - f(x_0,y_0) = f_y(x_0,y_0)(y-y_0)$

THEOREM 1. *An equation of the tangent plane to the graph of the function $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is*

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

CONCLUSION: A normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is

$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0$

$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

CONCLUSION: A normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is

$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0$

$$\mathbf{n} = \mathbf{n}(x_0, y_0) = \pm\langle f_x(x_0,y_0),\ f_y(x_0,y_0),\ -1 \rangle.$$

The line through the point $P(x_0, y_0, f(x_0, y_0))$ parallel to the vector $\mathbf{n}$ is perpendicular to the above tangent plane. This line is called **the normal line** to the surface $z = f(x, y)$ at P. It follows that this normal line can be expressed parametrically as

$$x = x_0 + f_x(x_0,y_0)\,t$$
$$y = y_0 + f_y(x_0,y_0)\,t$$
$$z = \underbrace{f(x_0,y_0)}_{z_0} - t$$

EXAMPLE 2. *Find an equation of the tangent plane to the graph of the function $z = x^2 + y^2 + 8$ at the point $(1, 1)$.*

$f(1,1) = 1^2 + 1^2 + 8 = 10$

$f_x = 2x \Rightarrow f_x(1,1) = 2$

$f_y = 2y \Rightarrow f_y(1,1) = 2$

$\Rightarrow$ The equation of the tangent plane is

$z - 10 = 2(x-1) + 2(y-1) \Leftrightarrow$

$2x + 2y - z = -10 + 2 + 2 = -6$

$\boxed{2x + 2y - z = -6} \Leftrightarrow z = 2x + 2y + 6$

EXAMPLE 3. *Find parametric equations for the normal line to the surface $z = \underbrace{e^{4y}\sin(4x)}_{f(x,y)}$ at the point $P(\pi/8, 0, 1)$*

$x_0 = \pi/8,\ y_0 = 0,\ z_0 = 1$

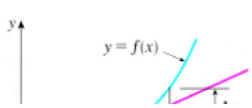
$z_x = f_x = 4e^{4y}\cos(4x) \Rightarrow f_x\left(\frac{\pi}{8}, 0\right) = 4e^0 \overbrace{\cos\left(\underbrace{4\cdot\frac{\pi}{8}}_{\frac{\pi}{2}}\right)}^{0} = 0$

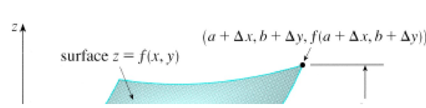
$z_y = f_y = 4e^{4y}\sin(4x) \Rightarrow f_y\left(\frac{\pi}{8}, 0\right) = 4\underbrace{e^0}_{1}\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = 4$

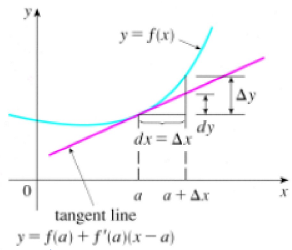
The normal vector to the tangent plane is $\langle 0, 4, -1\rangle \Rightarrow$ the parametric equation of the normal line is: $x = \underbrace{\frac{\pi}{8}}_{x_0 + 0\cdot t}$, $y = \underbrace{0 + 4t}_{y_0 + 4t} = 4t$, $z = \underbrace{1 - t}_{z_0 - t}$

Differentials. Given $z = f(x, y)$, denote $\Delta x = x - a$ and $\Delta y = y - b$ the increments of $x = a$ and $y = b$, respectively and by

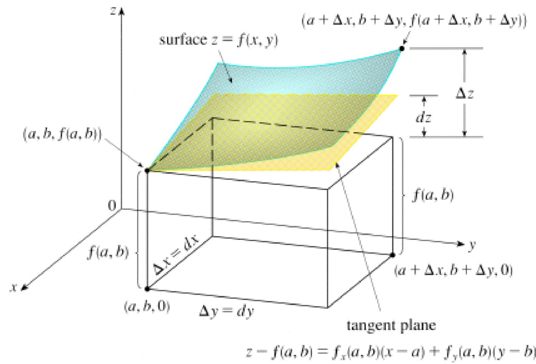
$$\Delta z = f(x, y) - f(a, b) = f(a + \Delta x, b + \Delta y) - f(a, b)$$







the best linear approximation of $f(x)$ near $x=a$



$$\underbrace{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}_{\text{the best linear approxim. of } f(x,y) \text{ near } (x,y)=(a,b)}$$

[1]the pictures are from our textbook

DEFINITION 4. *If $z = f(x, y)$, then f is differentiable at (a, b) if $\Delta z =$ can be expressed in the form*

$$\Delta z = \overbrace{f_x(a, b)\Delta x + f_y(a, b)\Delta y}^{\text{linear in } \Delta x \,\&\, \Delta y} + \underbrace{\varepsilon_1(x, y)\Delta x + \varepsilon_2(x, y)\Delta y}_{\text{remainder}},$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x$, ~~and~~ $\Delta y) \to (0, 0)$.

The **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables. The **differential** $\mathrm{d}z$ (or the **total differential**) is defined by

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y. = f_x(a,b)\,dx + f_y(a,b)\,dy$$

FACT: $\Delta z \approx \mathrm{d}z$.

This implies:

$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + \mathrm{d}z(a, b)$$

or

$$f(\underbrace{a+\Delta x}_{x}, \underbrace{b+\Delta y}_{y}) \approx f(a,b) + f_x(a,b)\underbrace{\Delta x}_{(x-a)} + f_y(a,b)\underbrace{\Delta y}_{x-b}$$

EXAMPLE 5. *Use differentials to find an approximate value for $\sqrt{1.03^2 + 1.98^3}$.*

$f(x,y) = \sqrt{x^2+y^3}$

$a = 1, \quad b = 2$

$\Delta x = \overbrace{1.03}^{x} - \overbrace{1}^{a} = 0.03$

$\Delta y = \underbrace{1.98}_{y} - \underbrace{2}_{} = -0.02$

$f(1,2) = \sqrt{1^2+2^3} = \sqrt{9} = 3$

$f_x = \frac{1}{\cancel{2}\sqrt{x^2+y^3}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^3}} \Rightarrow f_x(1,2) = \frac{1}{3}$

$$f_x = \frac{1}{\cancel{2}\sqrt{x^2+y^3}}\cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^3}} \Rightarrow f_x(1,2) = \frac{1}{3}$$

$$f_y = \frac{1}{2\sqrt{x^2+y^3}}\cdot 3y^2 = \frac{3y^2}{2\sqrt{x^2+y^3}} \Rightarrow f_y(1,2) = \frac{\cancel{3}\cdot 4}{2\cdot \cancel{3}} = 2$$

$$\sqrt{(1.03)^2+(1.98)^3} \approx \underbrace{f(1,2)}_{3} + \underbrace{\underbrace{f_x(1,2)}_{\frac{1}{3}}\,\underbrace{\Delta x}_{0.03}}_{0.01} + \underbrace{\underbrace{f_y(1,2)}_{2}\,\underbrace{\Delta y}_{(-0.02)}}_{-0.04} = 3+0.01-0.04 = \boxed{2.97}$$

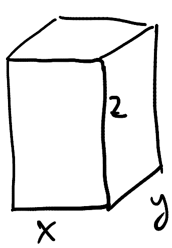
By calculator: 2.97040

If $u = f(x, y, z)$ then the differential $\mathrm{d}u$ at the point $(x, y, z) = (a, b, c)$ is defined in terms of the differentials $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ of the independent variables:

$$\mathrm{d}u(a, b, c) = f_x(a, b, c)\mathrm{d}x + f_y(a, b, c)\mathrm{d}y + f_z(a, b, c)\mathrm{d}z.$$

$$\Delta u \approx \underbrace{du}_{\text{a linear function in } dx,\, dy,\, dz}$$

EXAMPLE 6. *The dimensions of a closed rectangular box are measured as* 80 *cm,* 60 *cm and* 50 *cm, respectively, with a possible error of* 0.2 *cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.*



Surface area

$$S = 2(xz + yz + xy)$$

$$(a, b, c) = (80, 60, 50)$$

$$|\underbrace{\Delta x}_{dx}| \le 0.2,\quad |\underbrace{\Delta y}_{dy}| \le 0.2,\quad |\underbrace{\Delta z}_{dz}| \le 0.2$$

$$\underbrace{\Delta S(80,60,50)}_{\text{the error of measurement of } S} \approx dS(80,60,50) = S_x(80,60,50)\,dx + S_y(80,60,50)\,dy + S_z(80,60,50)\,dz$$

$$S = 2(xy + xz + yz)$$

$$S_x = 2(y+z) \Rightarrow S_x(80,60,50) = 220$$

$$|dS(80,60,50)| \overset{\text{triangle ineq.}}{\le} 220|dx| + 260|dy| + 280|dz| \le$$

$|a+b| \le |a|+|b|$

$$S_x = 2(y+z) \Rightarrow S_x(80,60,50) = 220$$
$$S_y = 2(x+z) \Rightarrow S_y(80,60,50) = 260$$
$$S_z = 2(x+y) \Rightarrow S_z(80,60,50) = 280$$

$$|dS(80,60,50)| \le$$
$$220\underbrace{|dx|}_{\le 0.2} + 260\underbrace{|dy|}_{\le 0.2} + 280\underbrace{|dz|}_{\le 0.2} \le$$
$$\le (\underbrace{220+260+280}_{760}) \cdot 0.2 = \boxed{152}\,cm^2$$

A function $f(x, y)$ is **differentiable** *at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).*

For example, all polynomial and rational functions are differentiable on their natural domains.

Let a surface S be a graph of a differentiable function f. As we zoom in toward a point on the surface S , the surface looks more and more like a plane (its tangent plane) and we can approximate the function f by a linear function of two variables.

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) =: L(x, y).$$