F19_LN_14_4

Thursday, September 26, 2019 10:31 PM

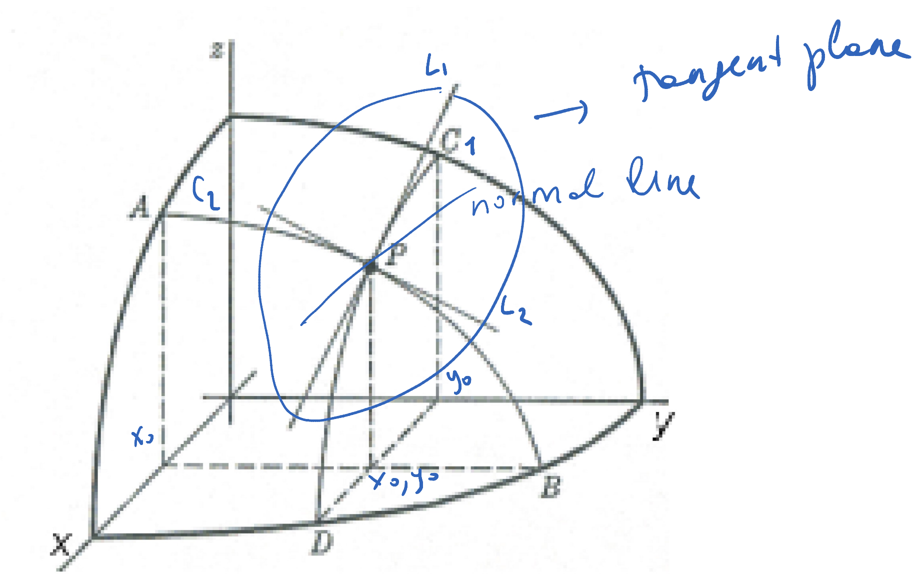
F19_LN_1...

14.4: Tangent Planes and Differentials

Suppose that $f(x, y)$ has continuous first partial derivatives and a surface S has equation $z = f(x, y)$. Let $P(x_0, y_0, z_0)$ be a point on S, i.e. $z_0 = f(x_0, y_0)$.

Denote by C_1 the trace to $f(x, y)$ for the plane $y = y_0$ and denote by C_2 the trace to $f(x, y)$ for the plane $x = x_0$. let L_1 be the tangent line to the trace C_1 and let L_2 be the tangent line to the trace C_2.

The **tangent plane** to the surface S (or to the graph of $f(x, y)$) at the point P is defined to be the plane that contains both the tangent lines L_1 and L_2.



THEOREM 1. *An equation of the tangent plane to the graph of the function $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is*

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

→ $y = y_0$ → equation of L_1

$x = x_0$ → equation of L_2

CONCLUSION: A normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is

$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)$ ...

$\to\ y = y_0 \to$ equation of ...

CONCLUSION: A normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ is

$x = x_0 \to$ equation of L_2

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) - (z - f(x_0,y_0)) = 0$$

$$\mathbf{n} = \mathbf{n}(x_0, y_0) = \pm\langle f_x(x_0,y_0),\ f_y(x_0,y_0),\ -1 \rangle.$$

The line through the point $P(x_0, y_0, f(x_0, y_0))$ parallel to the vector $\mathbf{n}$ is perpendicular to the above tangent plane. This line is called **the normal line** to the surface $z = f(x, y)$ at P. It follows that this normal line can be expressed parametrically as

$$\begin{cases} x = x_0 + f_x(x_0,y_0)\,t \\ y = y_0 + f_y(x_0,y_0)\,t \\ z = \underbrace{f(x_0,y_0)}_{z_0} - t \end{cases}$$

EXAMPLE 2. *Find an equation of the tangent plane to the graph of the function* $z = x^2 + y^2 + 8$ *at the point* $(1, 1)$.

$f(1,1) = 1^2 + 1^2 + 8 = 10$

$f_x = 2x \Rightarrow f_x(1,1) = 2$

$f_y = 2y \Rightarrow f_y(1,1) = 2$

$\Rightarrow$ The equation of the tangent plane is

$$\underbrace{z - 10}_{f(1,1)} = \underbrace{2}_{f_x(1,1)}(x - \underbrace{1}_{x_0}) + \underbrace{2}_{f_y(1,1)}(y - \underbrace{1}_{y_0})$$

$$\boxed{z = 2x + 2y + 6} \iff \boxed{2x - 2y - z = -6}$$

EXAMPLE 3. *Find parametric equations for the normal line to the surface* $z = e^{4y}\sin(4x)$ *at the point* $P(\underbrace{\pi/8}_{x_0}, \underbrace{0}_{y_0}, \underbrace{1}_{z_0})$

$(x_0,y_0) = (\frac{\pi}{8}, 0)$ $\qquad e^0 \sin(\underbrace{4\cdot\frac{\pi}{8}}_{\frac{\pi}{2}}) = 1$

$z_x = f_x = 4e^{4y}\cos(4x) \Rightarrow f_x(\frac{\pi}{8}, 0) = 4e^0 \underbrace{\cos(4\cdot\frac{\pi}{8})}_{\cos(\frac{\pi}{2})} = 0$

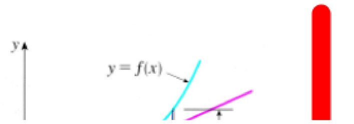
$z_y = f_y = 4e^{4y}\sin(4x) \Rightarrow f_y(\frac{\pi}{8}, 0) = 4\underbrace{e^0}_{1}\underbrace{\sin\frac{\pi}{2}}_{1} = 4$

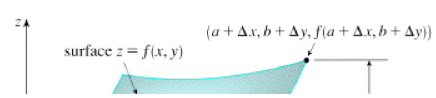
The parametric equation of the normal line is: $x = \frac{\pi}{8} + \underbrace{0\cdot t}_{0} = \frac{\pi}{8}$

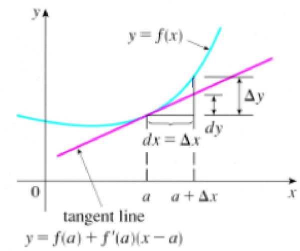
$y = 0 + 4t = 4t, \quad z = 1 - t$

Differentials. Given $z = f(x, y)$, denote $\Delta x = x - a$ and $\Delta y = y - b$ the increments of $x = a$ and $y = b$, respectively and by

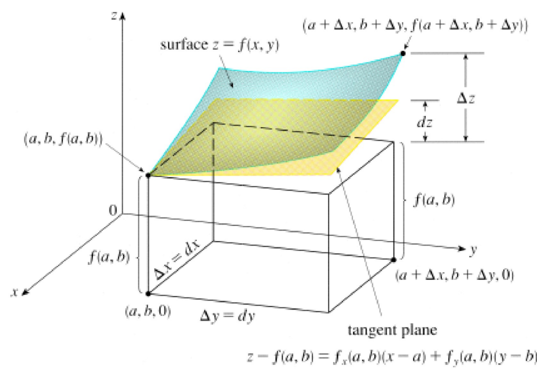
$$\Delta z = f(x, y) - f(a, b) = f(a + \Delta x, b + \Delta y) - f(a, b)$$







the best linear approximation of $f(x)$ near $x=a$



$z = \underbrace{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$ [1]

the best linear approximation of $f(x,y)$ near $(x,y)=(a,b)$

[1] the pictures are from our textbook

DEFINITION 4. *If $z = f(x, y)$, then f is differentiable at (a, b) if Δz can be expressed in the form*

$$\Delta z = \overbrace{f_x(a,b)\underbrace{\Delta x}_{x-a} + f_y(a,b)\underbrace{\Delta y}_{y-b}}^{\text{linear part} = dz} + \overbrace{\varepsilon_1(x,y)\Delta x + \varepsilon_2(x,y)\Delta y}^{\text{the remainder}},$$

which tends faster to 0 than the linear part

$\Delta z = f(x,y) - f(a,b)$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

The **differentials** dx and dy are independent variables. The **differential** dz (or the **total differential**) at (a,b) is defined by

$\Delta x = dx$
$\Delta y = dy$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. = \underbrace{f_x(a,b)\,dx + f_y(a,b)\,dy}_{\text{the linear function in } dx \,\&\, dy}$$

FACT: $\Delta z \approx dz$.

This implies:

$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + dz(a, b)$$

or

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + f_x(a,b)\underbrace{\Delta x}_{x-a} + f_y(a,b)\underbrace{\Delta y}_{y-b}$$

EXAMPLE 5. *Use differentials to find an approximate value for* $\sqrt{1.03^2 + 1.98^3}$.

$f(x,y) = \sqrt{x^2+y^3}$ $\qquad f(\underbrace{1.03}_{x}, \underbrace{1.98}_{y})$?

Take $a=1$, $b=2$

$f(1,2) = \sqrt{1^2+2^3} = \sqrt{9} = 3$, $\Delta x = x - a = 1.03 - 1 = 0.03$

$\Delta y = y - b = 1.98 - 2 = -0.02$

$f_x = \frac{1}{2\sqrt{x^2+y^3}} \cdot 2x = \frac{x}{\sqrt{x^2+y^3}} \Rightarrow f_x(1,2) = \frac{1}{3}$

$\Delta y = y - \ldots$

$$f_x = \frac{1}{\cancel{2}\sqrt{x^2+y^3}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^3}} \Rightarrow f_x(1,2) = \frac{1}{3}$$

$$f_y = \frac{1}{2\sqrt{x^2+y^3}} \cdot 3y^2 \Rightarrow f_y(1,2) = \frac{3\cdot 4}{2\cdot 3} = 2$$

$$\sqrt{(1.03)^2+(1.98)^3} = f(1.03, 1.98) \approx \underbrace{3}_{f(1,2)} + \underbrace{\frac{1}{3}}_{f_x(1,2)} \cdot \underbrace{0.03}_{\Delta x} + \underbrace{2}_{f_y(1,2)} \cdot \underbrace{(-0.02)}_{\Delta y} =$$

$$3 + 0.01 - 0.04 = 2.97 \Rightarrow \sqrt{(1.03)^2+(1.98)^3} \approx 2.97$$

By calculator: ≈ 2.97040

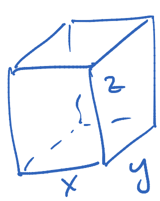
If $u = f(x, y, z)$ then the differential du at the point $(x, y, z) = (a, b, c)$ is defined in terms of the differentials dx, dy and dz of the independent variables:

$$du(a, b, c) = f_x(a, b, c)dx + f_y(a, b, c)dy + f_z(a, b, c)dz.$$

$$u(x,y,z) - u(a,b,c) \approx du(a,b,c)$$

$dx = x - a$
$dy = y - b$
$dz = z - c$

EXAMPLE 6. *The dimensions of a closed rectangular box are measured as* 80 *cm,* 60 *cm and* 50 *cm, respectively, with a possible error of* 0.2 *cm in each dimension. Use differentials to estimate the maximum error in calculating the surface area of the box.*



Surface area $S = 2(xz + yz + xy)$

$$\Delta S(80,60,50) \approx dS(80,60,50) =$$

$$= S_x(80,60,50)dx + S_y(80,60,50)dy + S_z(80,60,50)dz \quad (*)$$

$|\underbrace{dx}_{\Delta x}| \le 0.2$, $|dy| \le 0.2$, $|dz| \le 0.2$

$$S_x = 2(y+z), \quad S_y = 2(x+z), \quad S_z = 2(x+y)$$

triangle ineq. $|a+b| \le |a| + |b|$

$$\Delta S(80,60,50) \approx |dS(80,60,50)| \underset{(*)}{\le} |\underbrace{S_x(80,60,50)}_{>0}| \cdot \underbrace{|\Delta x|}_{\le 0.2} + |\underbrace{S_y(80,60,50)}_{>0}| \underbrace{|dy|}_{\le 0.2} + |\underbrace{S_z(80,60,50)}_{>0}| \underbrace{|dz|}_{\le 0.2} \le$$

$$\le \underbrace{(S_x + S_y + S_z)}_{4(x+y+z)}\Big|_{(x,y,z)=(80,60,50)} \cdot 0.2 = 4(\underbrace{80+60+50}_{190}) \cdot 0.2 = \boxed{152 \text{ cm}^2}$$

$$\le \left(\underbrace{S_x + S_y + S_z}_{4(x+y+z)}\right)\Big|_{(x,y,z)=(80,60,50)} \cdot 0.2 = 4(\underbrace{80+60+50}_{190}) \cdot 0.2 = \boxed{152 \text{ cm}}$$

$$S_x + S_y + S_z = 2y + 2z + 2x + 2z + 2x + 2y$$

A function $f(x, y)$ is **differentiable** *at (a, b) if its partial derivatives f_x and f_y exist and are continuous at (a, b).*

For example, all polynomial and rational functions are differentiable on their natural domains.

Let a surface S be a graph of a differentiable function f. As we zoom in toward a point on the surface S , the surface looks more and more like a plane (its tangent plane) and we can approximate the function f by a linear function of two variables.

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) =: L(x, y).$$