F19_LN_14_5

Monday, September 30, 2019 12:43 AM

14.5: The Chain Rule

$y = \underbrace{f(g(t))}_{\text{composition}}$

Chain Rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dg}{dt}. = \frac{dy}{dx}\frac{dx}{dt}$$

EXAMPLE 1. *Let $z = x^y$, where $x = t^2$, $y = \sin t$. Compute $z'(t)$.* $t > 0$

From Calc 1

$$z = (t^2)^{\sin t} = \underbrace{t}_{e^{\ln t}}{}^{2\sin t} = e^{2\ln t \sin t} \Rightarrow$$

$x = g(t) = 2\ln(t)\sin t$

$z = f(x) = e^x$

$$\frac{dz}{dt} = \underbrace{\frac{dz}{dx}}_{e^x}\cdot\underbrace{\frac{dx}{dt}}_{\frac{2}{t}\sin t + 2\ln(t)\cos t} = \underbrace{e^{\overbrace{2\ln t\sin t}^{x}}}_{t^{2\sin t}}\left(\frac{2}{t}\sin t + 2\ln t\cos t\right) = (t^2)^{\sin t}\left(\frac{2}{t}\sin t + 2\ln t\cos t\right)$$

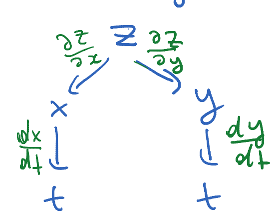
`Assume that all functions below have continuous derivatives (ordinary or partial).`

- CASE 1: $z = f(x, y)$, where $x = x(t)$, $y = y(t)$ and compute $z'(t)$.

  $z(t) = f(x(t), y(t))$

  Chain Rule:
  $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Tree diagram



Proof

at the point $(x_0, y_0) = (x(t_0), y(t_0))$

$$\frac{\Delta z}{\Delta t} = z_x\frac{\Delta x}{\Delta t} + z_y\frac{\Delta y}{\Delta t} + \epsilon_1(x, y)\frac{\Delta x}{\Delta t} + \epsilon_2(x, y)\frac{\Delta y}{\Delta t}$$

$\Delta t \to 0$: $\quad \frac{dz}{dt} = z_x\frac{dx}{dt} + z_y\frac{dy}{dt}$, $\quad \epsilon_1, \epsilon_2 \to 0$

SOLUTION OF EXAMPLE 1:

Using Calc 3 $\quad z = x^y$, $x(t) = t^2$, $y(t) = \sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t) = y x^{y-1}\cdot 2t + \ln x\cdot x^y\cos t$$

$(x^n)' = n x^{n-1}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^y) = y x^{y-1} \qquad x'(t) = 2t$$

$(\underbrace{a^x}_{e^{\ln a\, x}})' = \ln a\cdot a^x$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^y) = \ln x\cdot x^y \qquad y'(t) = \cos t$$

$(a^x)' = \ln a \cdot a^x$
$e^{\ln a x}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^y) = \ln x \cdot x^y \qquad y'(t) = \cos t$$

$$\underset{\text{substitute } x(t) \text{ \& } y(t)}{=} \sin t \cdot \underbrace{(t^2)^{\sin t - 1} \cdot 2t}_{\substack{2t^{2\sin t - 2 + 1} = 2t^{2\sin t - 1} \\ 2(t^2)^{\sin t} \cdot \frac{1}{t}}} + \underbrace{\ln t^2}_{2\ln t} \cdot (t^2)^{\sin y} \cdot \cos t = (t^2)^{\sin t}\left(\frac{2}{t}\sin t + 2\ln t \cos t\right)$$

the same as in ex. 1

EXAMPLE 2. *The radius of a right circular cone is increasing at a rate of* 1.8 *cm/s while its height is decreasing at a rate* 2.5 *cm/s. At what rate is the volume of the cone changing when the radius is* 120 *cm and the height is* 140 *cm.*

$$\underset{\text{volume}}{V(r,h)} = \frac{1}{3}\pi r^2 h$$

$$r = r(t), \quad h = h(t), \quad r_0 = 120, \quad h_0 = 140$$
$$r_0 = r(t_0) \quad h_0 = h(t_0)$$

This is what we want to find

$$\frac{dr}{dt}(t_0) = 1.8, \quad \frac{dh}{dt}(t_0) = -2.5$$

$$\frac{d}{dt}V(t_0) = \frac{\partial V}{\partial r}(r_0,h_0)\frac{dr}{dt}(t_0) + \frac{\partial V}{\partial h}(r_0,h_0)\frac{dh}{dt}(t_0) = (*)$$

Multiver. chain rule

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h \Rightarrow \frac{\partial V}{\partial r}(120, 140) = \frac{2}{3}\pi\, 120 \cdot 140$$

$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2 \Rightarrow \frac{\partial V}{\partial h}(120, 140) = \frac{1}{3}\pi\, 120^2$$

Plug into $(*)$: $\frac{dV}{dt} = \frac{2}{3}\pi\, 120 \cdot 140 \cdot \underbrace{1.8}_{\frac{dr}{dt}} + \frac{1}{3}\pi\, 120^2 \cdot \underbrace{(-2.5)}_{\frac{dh}{dt}} =$

$$= \frac{1}{3}\pi\, 1200\Big(\underbrace{14 \cdot 3.6}_{50.4} - \underbrace{12 \cdot 2.5}_{6.5 = 30}\Big) = 400\pi \cdot 20.4 = 8160\pi$$

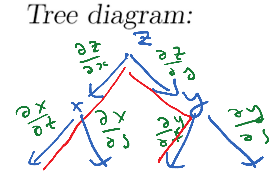
- CASE 2: $z = f(x,y)$, where $x = x(s,t)$, $y = y(s,t)$ and compute z_s and z_t.

Chain Rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

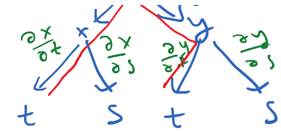
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tree diagram:



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

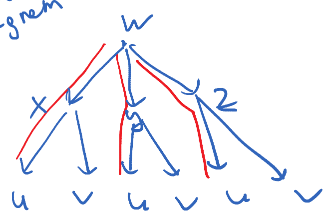


— the paths from z to t

EXAMPLE 3. *Write out the Chain Rule for the case where* $w = f(x, y, z)$ *and* $x = x(u, v)$, $y = y(u, v)$ *and* $z = z(u, v)$.

$w = f(x(u,v), y(u,v), z(u,v))$

Tree diagram

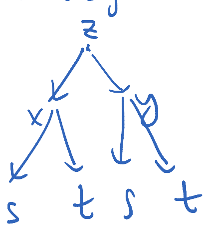


$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

compared to the first formula u is replaced by v everywhere

Tree diagram



EXAMPLE 4. *If* $z = \sin x \cos y$, *where* $x = (s-t)^2$, $y = s^2 - t^2$ *find* $z_s + z_t$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \underbrace{\cos x \cos y}_{\frac{\partial z}{\partial x}}\cdot\underbrace{2(s-t)}_{\frac{\partial x}{\partial s}} + \underbrace{(-\sin x \sin y)}_{\frac{\partial z}{\partial y}}\cdot\underbrace{2s}_{\frac{\partial y}{\partial s}}$$

$+$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \cos x \cos y \cdot 2(s-t)\cdot(-1) + (-\sin x \sin y)\cdot(-2t)$$

$$z_s + z_t = -\sin x \sin y\,(2s-2t) \underset{\substack{x=(s-t)^2\\ y=s^2-t^2}}{=} 2(t-s)\sin\left((s-t)^2\right)\sin(s^2-t^2)$$

EXAMPLE 5. *Show that*

$$g(s,t) = f(\underbrace{s^2 - t^2}_{x}, \underbrace{t^2 - s^2}_{y})$$

satisfies the equation

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satisfies the equation

Set $x = s^2-t^2$, $y = t^2-s^2$

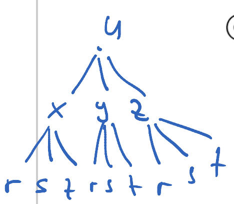
$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$ — linear PDE of first order.

$$t \times \frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\cdot 2s + \frac{\partial f}{\partial y}\cdot(-2s) \quad \times t$$

$$+$$

$$s \times \frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}\cdot(-2t) + \frac{\partial f}{\partial y}\cdot(2t) \quad \times s$$

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \cancel{\frac{\partial f}{\partial x}\cdot 2st} + \cancel{\frac{\partial f}{\partial y}\cdot(-2st)} + \cancel{\frac{\partial f}{\partial x}\cdot(-2ts)} + \cancel{\frac{\partial f}{\partial y}\cdot 2ts} = 0$$



EXAMPLE 6. *If* $u = x^2y + y^3z^2$ *where* $x = rse^t$, $y = r + s^2e^{-t}$, $z = rs\sin t$, *find* u_s *when* $(r,s,t) = (1,2,0)$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s} = 2xy\, re^t +$$

$$+ (x^2 + 3y^2z^2)\, 2se^{-t} + 2y^3z\cdot r\sin t = (*)$$

$\frac{\partial u}{\partial s}(1,2,0)$? $(r,s,t) = (1,2,0) \Rightarrow$

$$(x,y,z) = (rse^t, r+s^2e^{-t}, rs\sin t)\Big|_{(r,s,t)=(1,2,0)} =$$

$$= (1\cdot 2\cdot \underbrace{e^0}_{1}, 1 + 2^2\cdot\underbrace{e^{-0}}_{1}, 1\cdot 2\underbrace{\sin 0}_{0}) = (2,5,0)$$

$$\frac{\partial u}{\partial s}(1,2,0) = 2\cdot 2\cdot 5\cdot 1\underbrace{e^0}_{1} + (2^2 + \underbrace{3\cdot 5^2\cdot 0}_{0})\, 2\cdot 2\cdot\underbrace{e^{-0}}_{1} + \underbrace{2\cdot 5^3\cdot 0\cdot 1\sin 0}_{0}$$

plug in $(*)$
$(r,s,t) = (1,2,0)$ & $(x,y,z) = (2,5,0)$

$$= 20 + 16 = \boxed{36}$$

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Implicit differentiation: Suppose that an equation

$$F(x,y)=0$$

Example $x^2+y^2-1=0$

defines y implicitly as a differentiable function of x, i.e. $y=y(x)$, where $F(x,y(x))=0$ for all x in the domain of $y(x)$. Find y':

$$F(x,y(x))=0 \Rightarrow \frac{d}{dx}F(x,y(x)) = \frac{d}{dx}0 = 0$$

On the other hand, we can use the chain rule:

$$\frac{d}{dx}F(x,y(x)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

If $\frac{\partial F}{\partial y} \neq 0$, then $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$

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EXAMPLE 7. *Find* y' *if* $x^4 + y^3 = 6e^{xy}$. $\Leftrightarrow$ $x^4 + y^3 - 6e^{xy} = 0$

$F(x,y) = x^4 + y^3 - 6e^{xy}$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \boxed{-\frac{4x^3 - 6ye^{xy}}{3y^2 - 6xe^{xy}}}$$

Suppose that an equation

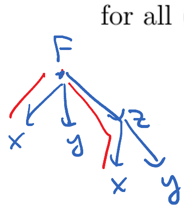
$$F(x, y, z) = 0$$

defines z implicitly as a differentiable function of x and y, i.e. $z = z(x, y)$, where

$$\underbrace{F(x, y, z(x, y))}_{g(x,y)} = 0$$

for all (x, y) in the domain of z. Find the partial derivatives z_x and z_y:

Assume that $F_z \neq 0$



$$0 = \frac{\partial g}{\partial x} \underset{\text{Chain rule}}{=} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$0 = \frac{\partial g}{\partial y} \underset{\text{Chain rule}}{=} \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

EXAMPLE 8. *If* $x^4 + y^3 + z^2 + xye^z = 10$ *find*

(a) z_x *and* z_y $F = x^4 + y^3 + z^2 + xye^z - 10$

$$z_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{4x^3 + ye^z}{2z + xye^z}$$

$$z_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{3y^2 + xe^z}{2z + xye^z}$$

$$z_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{\quad}{2z + xye^z}$$

(b) x_y *and* x_z

$$\frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{3y^2 + xe^z}{4x^3 + ye^z}$$

$$\frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = \ldots$$