F19_LN_14_5

Monday, September 30, 2019 12:44 AM

14.5: The Chain Rule

composition $y = f(g(t))$

Chain Rule for functions of a single variable: If $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dg}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

$x_0 = g(t_0)$

$$\left.\frac{dy}{dt}\right|_{t=t_0} = \left.\frac{dy}{dx}\right|_{x=x_0} \cdot \left.\frac{dg}{dt}\right|_{t=t_0}$$

EXAMPLE 1. *Let $z = x^y$, where $x = t^2$, $y = \sin t$. Compute $z'(t)$.*

$t>0$

Calc 1 solution:

$z = \underset{\text{plug } x(t) \,\&\, y(t)}{(t^2)^{\sin t}} = \underset{t = e^{\ln t}}{t^{2\sin t}} = e^{2\ln t \sin t}$

$z = e^u$, when $u = 2\ln t \sin t$

$$\frac{dz}{dt} = \underbrace{\frac{dz}{du}}_{e^u} \cdot \underbrace{\frac{du}{dt}}_{\frac{2}{t}\cdot\sin t + 2\ln t\cdot\cos t} =$$

$$= e^u\left(\frac{2}{t}\sin t + 2\ln t \cos t\right) = \underbrace{e^{2\ln t\sin t}}_{(t^2)^{\sin t}}\left(\frac{2}{t}\sin t + 2\ln t\cdot\cos t\right)$$

Assume that all functions below have continuous derivatives (ordinary or partial).

$z(t) = f(x(t), y(t))$

- CASE 1: $z = f(x, y)$, where $x = x(t)$, $y = y(t)$ and compute $z'(t)$.

  Chain Rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$z(t) = f(x(t), y(t))$; f_x, x_t, f_y, y_t

Tree diagram

$z \xrightarrow{\partial z/\partial x} x \xrightarrow{dx/dt} t$, $\quad z \xrightarrow{\partial z/\partial y} y \xrightarrow{dy/dt} t$

Proof: $\displaystyle \underset{\downarrow t\to t_0 \;\; z'(t_0)}{\frac{\Delta z}{\Delta t}} = \frac{\partial z}{\partial x}(x_0,y_0)\underset{\downarrow t\to t_0\;\; x'(t_0)}{\frac{\Delta x}{\Delta t}} + \frac{\partial z}{\partial y}(x_0,y_0)\underset{\downarrow t\to t_0\;\; y'(t_0)}{\frac{\Delta y}{\Delta t}} + \varepsilon_1(x,y)\frac{\Delta x}{\Delta t} + \varepsilon_2(x,y)\frac{\Delta y}{\Delta t}$, $\quad \varepsilon_1, \varepsilon_2 \to 0$

$x_0 = x(t_0)$, $y_0 = y(t_0)$

$$\left.\frac{dz}{dt}\right|_{t=t_0} = \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(x_0,y_0)} \cdot \left.\frac{dx}{dt}\right|_{t=t_0} + \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(x_0,y_0)} \cdot \left.\frac{dy}{dt}\right|_{t=t_0}$$

SOLUTION OF EXAMPLE 1:

Table: $(x^n)' = nx^{n-1}$, $(e^x)' = \ln e \cdot e^x$, $(e^{\ln a \cdot x})'$

$z = x^y$, $x = t^2$, $y = \sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\underbrace{x'(t)}_{2t} + \frac{\partial z}{\partial y}\underbrace{y'(t)}_{\cos t} = yx^{y-1}\cdot 2t + \ln x \cdot x^y \cdot \cos t =$$

$x = t^2$, $y = \sin t$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^y) = yx^{y-1}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^y) = \ln x \cdot x^y$$

$t^{2\sin t}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^y) = \ln x \cdot x^y \qquad x = t^2,\ y = \sin t$$

$$= \sin t \cdot \underbrace{(t^2)^{\sin t - 1} \cdot 2t}_{2t^{2\sin t - 2 + 1} = 2t^{2\sin t - 1} = 2t^{2\sin t}\cdot\frac{1}{t}} + \overbrace{\ln t^2}^{2\ln t} \cdot \overbrace{(t^2)^{\sin t}}^{t^{2\sin t}} \cdot \cos t = t^{2\sin t}\left(\frac{2}{t}\sin t + 2\ln t \cdot \cos t\right),$$

which coinsides with the previous answer

EXAMPLE 2. *The radius of a right circular cone is increasing at a rate of* 1.8 *cm/s while its height is decreasing at a rate* 2.5 *cm/s. At what rate is the volume of the cone changing when the radius is* 120 *cm and the height is* 140 *cm.*

$$\underbrace{V}_{\text{the volume of the cone}}(r,h) = \frac{1}{3}\pi r^2 h$$

$r = r(t)$, $h = h(t)$

$r_0 = 120$, $h_0 = 140$

$r_0 = r(t_0)$, $h_0 = h(t_0)$

Given: $\frac{dr}{dt}(t_0) = 1.8$, $\frac{dh}{dt}(t_0) = -2.5$

We are asked to find $\left.\frac{dV}{dt}\right|_{t=t_0}$

$$\left.\frac{dV}{dt}\right|_{t=t_0} = \frac{\partial V}{\partial r}(120,140)\cdot\underbrace{\frac{dr}{dt}(t_0)}_{1.8} + \frac{\partial V}{\partial h}(120,140)\underbrace{\frac{dh}{dt}(t_0)}_{-2.5} =$$

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h \Rightarrow \frac{\partial V}{\partial r}(120,140) = \frac{2}{3}\pi\, 120\cdot 140$$

$$\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2 \Rightarrow \frac{\partial V}{\partial h}(120,140) = \frac{1}{3}\pi\, 120^2$$

$$= \frac{2}{3}\pi\, 120\cdot 140\cdot 1.8 - \frac{1}{3}\pi\, 120^2\cdot 2.5 = \frac{1}{3}\pi\cdot\underbrace{120\cdot 10}_{1200}\Big(\underbrace{14\cdot 3.6}_{50.4} - \underbrace{12\cdot 2.5}_{6.5=30}\Big) =$$

$$= 400\pi\cdot 20.4 = \boxed{8160\pi}$$

- CASE 2: $z = f(x,y)$, where $x = x(s,t)$, $y = y(s,t)$ and compute z_s and z_t.

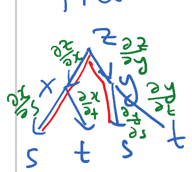
Tree diagram

Chain Rule:

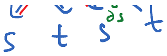
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Tree diagram:



there are

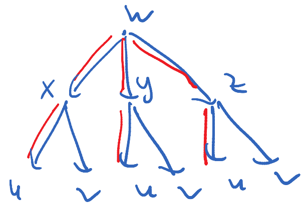


$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

there are
2 paths to arrive from z to s

EXAMPLE 3. *Write out the Chain Rule for the case where $w = f(x, y, z)$ and $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$.*

$w = f(x(u,v), y(u,v), z(u,v))$

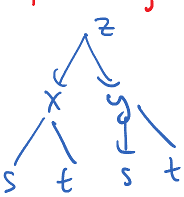


$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \quad (1)$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} \quad (2)$$

replace
u by v in (1)

Tree diagram



EXAMPLE 4. *If $z = \sin x \cos y$, where $x = (s - t)^2$, $y = s^2 - t^2$ find $z_s + z_t$.*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \underbrace{\cos x \cos y}_{\frac{\partial z}{\partial x}}\cdot\underbrace{2(s-t)}_{\frac{\partial x}{\partial s}} + $$

$$+\underbrace{(-\sin x \sin y)}_{\frac{\partial z}{\partial y}}\cdot\underbrace{2s}_{\frac{\partial y}{\partial s}}$$

$+$ → when we sum up

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \cos x \cos y\cdot 2(s-t)\cdot(-1) +$$

$$+ (-\sin x \sin y)\cdot(-2t)$$

$$\frac{\partial z}{\partial s} + \frac{\partial z}{\partial t} = -\sin x \sin y\,(2s - 2t) \underset{\substack{x=(s-t)^2 \\ y = s^2-t^2}}{=} 2(t-s)\sin[(s-t)^2]\sin(s^2-t^2)$$

EXAMPLE 5. *Show that*

$$g(s, t) = f(\underbrace{s^2 - t^2}_{x}, \underbrace{t^2 - s^2}_{y})$$

satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0. \rightarrow \text{linear } 1^{st} \text{ order PDE}$$

satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0. \to$$ linear 1st order PDE
partial differential equation

$x = s^2 - t^2$
$y = t^2 - s^2$

$$t \times \quad \frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\cdot 2s + \frac{\partial f}{\partial y}\cdot(-2s) \quad \begin{matrix}\times t\\ + \\ \times s\end{matrix}$$

$$+ \quad s \times \quad \frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}\cdot(-2t) + \frac{\partial f}{\partial y}\cdot(2t)$$

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \cancel{\frac{\partial f}{\partial x}\cdot 2st} - \cancel{\frac{\partial f}{\partial y}2st} - \cancel{\frac{\partial f}{\partial x}}2st + \cancel{\frac{\partial f}{\partial y}}\cdot 2ts = 0$$

EXAMPLE 6. *If* $u = x^2y + y^3z^2$ *where* $x = rse^t$, $y = r + s^2e^{-t}$, $z = rs\sin t$, *find* u_s *when* $(r,s,t) = (1,2,0)$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s} = 2xy\, re^t +$$

$$+ (x^2 + 3y^2z^2)\cdot 2se^{-t} + 2y^3z\cdot rs\sin t$$

$\frac{\partial u}{\partial s}(1,2,0)$? If $(r,s,t) = (1,2,0) \Leftrightarrow r=1, s=2, t=0$, then

$$(x,y,z) = (rse^t,\ r+s^2e^{-t},\ rs\sin t)\Big|_{(r,s,t)=(1,2,0)}$$

$$= (1\cdot 2\cdot \underbrace{e^0}_{1},\ 1 + 2^2\underbrace{e^{-0}}_{1},\ 1\cdot 2\cdot \underbrace{\sin 0}_{0}) = (2, 5, 0) \to \begin{matrix} x=2\\ y=5\\ z=0\end{matrix}$$

$$\frac{\partial u}{\partial s}(1,2,0) = 2\cdot 2\cdot 5\cdot 1\cdot\underbrace{e^0}_{1} + (2^2 + \underbrace{3\cdot 5^2\cdot 0^2}_{0})\cdot 2\cdot 2\underbrace{e^{-0}}_{1} +$$

$$+ \underbrace{2\cdot 5^3\cdot 0\cdot 1\cdot \sin 0}_{0} = 20 + 16 = \boxed{36}$$

Implicit Function Thm

If $\boxed{\frac{\partial F}{\partial y} \neq 0}$ then locally one can solve this equation for y as a function of x

Implicit differentiation: Suppose that an equation

$$F(x, y) = 0$$

defines y implicitly as a differentiable function of x, i.e. $y = y(x)$, where $F(x, y(x)) = 0$ for all x in the domain of $y(x)$. Find y':

Tree diagram

$$\begin{tikzcd} & F \arrow[dl, "\frac{\partial F}{\partial x}"'] \arrow[dr, "\frac{\partial F}{\partial y}"] & \\ x & & y \arrow[d, "\frac{dy}{dx}"] \\ & & x \end{tikzcd}$$

$$F(x, y(x)) = 0 \Rightarrow \underbrace{\frac{d}{dx} F(x, y(x))}_{\downarrow \text{ chain rule}} = \frac{d}{dx} 0 = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0 \Rightarrow$$

If $\frac{\partial F}{\partial y} \neq 0$ then $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$, so $\frac{dy}{dx} \cancel{=} -\frac{F_x}{F_y}$

EXAMPLE 7. *Find y' if $x^4 + y^3 = 6e^{xy}$.* $\Leftrightarrow$ $x^4 + y^3 - 6e^{xy} = 0$

EXAMPLE 7. *Find y' if $x^4 + y^3 = 6e^{xy}$.* $\Leftrightarrow \; x^4 + y^3 - 6e^{xy} = 0$

Let $F(x,y) = x^4 + y^3 - 6e^{xy}$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x^3 - 6ye^{xy}}{3y^2 - 6xe^{xy}}$$

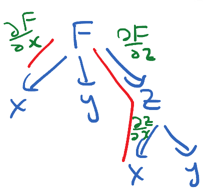
Suppose that an equation

$$F(x, y, z) = 0$$

defines z implicitly as a differentiable function of x and y, i.e. $z = z(x, y)$, where

$$\underbrace{F(x, y, z(x, y))}_{g(x,y)} = 0$$

for all (x, y) in the domain of z. Find the partial derivatives z_x and z_y:



$\frac{\partial}{\partial x} g(x,y) = 0$ chain rule $\Rightarrow$ $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \Rightarrow$ If $\frac{\partial F}{\partial z} \neq 0$ $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$

$\frac{\partial}{\partial y} g(x,y) = 0 \Rightarrow \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

EXAMPLE 8. *If $x^4 + y^3 + z^2 + xye^z = 10$ find*

(a) z_x *and* z_y

$F = x^4 + y^3 + z^2 + xye^z - 10$

$$z_x = \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{4x^3 + ye^z}{2z + xye^z}$$

$$z_y = \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + xe^z}{2z + xye^z}$$

(b) x_y *and* x_z

(b) x_y *and* x_z

$$x_y = \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} = -\frac{3y^2 + xe^z}{4x^3 + ye^z}$$

$$x_z = \frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = \frac{2z + xye^z}{4x^3 + ye^z}$$