# F19_LN_14_6

Monday, September 30, 2019 7:09 PM

F19_LN_1...

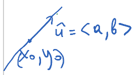


## 14.6: Directional Derivatives and the Gradient Vector

Recall that the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y)$ represent the rate of change of $f$ as we vary $x$ (holding $y$ fixed) and as we vary $y$ (holding $x$ fixed) respectively. In other words, $f_x(x,y)$ and $f_y(x,y)$ represent the rate of change of $f$ in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ respectively. Let's consider how to find the rate of change of $f$ if we allow both $x$ and $y$ to change simultaneously, or how to find the rate of change of $f$ in the direction of an arbitrary vector $\mathbf{u}$.

DEFINITION 1. *The rate of change of $f(x,y)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is called the* **directional derivative** *and it is denoted by $D_{\mathbf{u}}f(x,y)$.*

*The* **directional derivative** *of $f$ at $(x_0,y_0)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a,b\rangle$ is*




$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb)-f(x_0,y_0)}{h} = \frac{d}{dh} f(x_0+a\underbrace{h}_{\text{varies}}, y_0+bh)\Big|_{h=0}$$

*if this limit exists.*

REMARK 2. By comparing the last definition with the definitions of the partial derivatives:

$$f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}, \qquad f_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

$a = 1 \quad b = 0 \Rightarrow \hat{u} = \hat{i}$ $\qquad a = 0 \quad b = 1 \quad \hat{u} = \hat{j}$

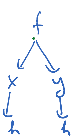
we see that

$$f_x(x_0,y_0) = D_{\hat{i}} f(x_0,y_0) \qquad \text{and} \qquad f_y(x_0,y_0) = D_{\hat{j}} f(x_0,y_0)$$

For computational purposes use the following theorem.

THEOREM 3. *If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any <u>unit</u> vector $\hat{\mathbf{u}} = \langle a,b\rangle$ and*

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b. = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle$$

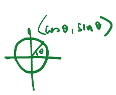



*Proof*

$$D_u f(x_0,y_0) = \frac{d}{dh} f(\underbrace{x_0+ah}_{x(h)}, \underbrace{y_0+bh}_{y(h)})\Big|_{h=0} = \frac{\partial f}{\partial x}(x_0,h_0)\cdot\underbrace{\frac{\partial x}{\partial h}}_{a} +$$

$$+ \frac{\partial f}{\partial y}(x_0,h_0)\underbrace{\frac{\partial y}{\partial h}}_{b} =$$

EXAMPLE 4. *Find the rate of change $f(x,y) = x^3 + \sin(xy)$ at the point $(1, \pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.*




$$\hat{u} = \langle \cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}\rangle = \langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle$$

$$f_x = 3x^2 + y\cos(xy) \Rightarrow f_x(1,\tfrac{\pi}{2}) = 3 + \tfrac{\pi}{2}\underbrace{\cos\tfrac{\pi}{2}}_{0} = 3$$

$$f_y = x\cos(xy) \Rightarrow f_y(1,\tfrac{\pi}{2}) = 1\cdot\cos\tfrac{\pi}{2} = 0$$

$$D_{\hat{u}} f(1,\tfrac{\pi}{2}) = f_x a + f_y b = 3\cdot\frac{1}{\sqrt{2}} + 0\cdot\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

***The Directional Derivative As The Dot Product Of Two Vectors. Gradient.***

DEFINITION 5. *The* **gradient** *of* $f(x,y)$ *is the vector function* $\nabla f$ *defined by*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Notations for gradient: $\mathbf{grad} f$ or $\nabla f$ which is read "del $f$". nable

EXAMPLE 6. *Find the gradient of* $f = \cos(xy) + e^x$ *at (0,3).*

$f_x = -y \sin xy + e^x \Rightarrow f_x(0,3) = -3\sin 0 + e^0 = 1$

$f_y = -x \sin xy \Rightarrow f_y(0,3) = 0$

$\nabla f(0,3) = \langle 1, 0\rangle = \hat{i}$

By Theorem 3 we have:

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle =$$
$$= \text{grad} f(x,y) \cdot \langle a, b\rangle = \text{grad} f(x,y) \cdot \hat{u}$$

`Formula for the directional derivative using the gradient vector:`

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \underbrace{\hat{\mathbf{u}}}_{\text{unit}}.$$

EXAMPLE 7. *Find the directional derivative for* $f$ *from Example 6 at* $(0,3)$ *in the direction* $\vec{u} = \langle 3,4\rangle$.

$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$

$$\hat{u} = \frac{\langle 3,4\rangle}{\sqrt{\underbrace{3^2+4^2}_{25}}} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$D_{\vec{u}} f(0,3) = \nabla f(0,3)\cdot \hat{u} = \langle 1, 0\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \frac{3}{5}$$

**The directional derivative of function of *three* variables**

THEOREM 8. *If* $f$ *is a differentiable function of* $x$, $y$ *and* $z$, *then* $f$ *has a directional derivative in the direction of any unit vector* $\hat{\mathbf{u}} = \langle a, b, c\rangle$ *and*

$$D_{\mathbf{u}} f(x,y) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \nabla f \cdot \hat{\mathbf{u}},$$

*where*

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

*is the gradient vector of* $f(x,y,z)$.

EXAMPLE 9. *Find the directional derivative of* $f(x, y, z) = z^3 - x^2y$ *at the point* $(1, 6, 2)$ *in the direction* $\mathbf{u} = \langle 1, -2, 3\rangle$.

$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$ $\quad |\vec{u}| = \sqrt{\underbrace{1^2+(-2)^2+3^2}_{14}} = \sqrt{14}$

$f_x = -2xy \Rightarrow f_x(1,6,2) = -2\cdot 1\cdot 6 = -12$

$f_y = -x^2 \Rightarrow f_y(1,6,2) = -1$ $\quad \Rightarrow \quad \nabla f(1,6,2) = \langle -12,-1,12\rangle$

$f_z = 3z^2 \Rightarrow f_z(1,6,2) = 3\cdot 2^2 = 12$

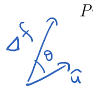
$D_{\vec{u}} f(1,6,2) = \nabla f(1,6,2)\cdot \hat{u} = \langle -12,-1,12\rangle \cdot \frac{1}{\sqrt{14}}\langle 1,-2,3\rangle =$

$= \frac{1}{\sqrt{14}}\left(-12\cdot 1 + (-1)\cdot(-2) + 12\cdot 3\right) = \boxed{\frac{26}{\sqrt{14}}}$

*QUESTION: In which of all possible directions does* $f$ *change fastest and what is the maximum rate of change.*

*ANSWER* is provided by the following theorem:

THEOREM 10. *Suppose* $f$ *is a differentiable function of two or three variables. The maximum value of the directional derivative* $D_{\mathbf{u}}f$ *is* $|\nabla f|$ *and it occurs when* $\mathbf{u}$ *has the same direction as the gradient vector* $\nabla f$.




*Proof.* $D_u f = \nabla f \cdot \hat{u} = |\nabla f|\cdot \underbrace{|\hat{u}|}_{1}\, \underbrace{\cos\theta}_{\le 1} \le |\nabla f|$

When the equality holds? $\iff \cos\theta = 1 \iff \theta = 0$

$\iff \hat{u}$ is in the direction of $\nabla f$ $\square$

EXAMPLE 11. *Suppose that the temperature at a point $(x, y, z)$ in the space is given by*

$$T(x,y,z) = \frac{100}{1+x^2+y^2+z^2}, \quad = 100\,(1+x^2+y^2+z^2)^{-1}$$

*where $T$ is measured in $^\circ C$ and $x, y, z$ in meters.*

**(a)** *In which direction does the temperature increase fastest at the point $(1,1,-1)$?*

It is in the direction of $\nabla T(1,1,-1)$

$$T_x = -100\,(1+x^2+y^2+z^2)^{-2}\cdot 2x = -\frac{200x}{(1+x^2+y^2+z^2)^2} \Rightarrow T_x(1,1,-1) = -\frac{200}{4^2} = -\frac{50}{4} = -\frac{25}{2}$$

Similarly $T_y = -\dfrac{200y}{(1+x^2+y^2+z^2)^2} \Rightarrow T_y\big|_{(1,1,-1)} = -\dfrac{25}{2}$

$$T_z = -\frac{200z}{(1+x^2+y^2+z^2)^2} \Rightarrow T_z\big|_{(1,1,-1)} = \frac{25}{2}$$

$$\nabla T(1,1,-1) = \left\langle -\frac{25}{2}, -\frac{25}{2}, \frac{25}{2}\right\rangle = \frac{25}{2}\langle -1,-1,1\rangle$$

**(b)** *What is the maximum rate of increase?*

It is $|\nabla T(1,1,-1)| = \dfrac{25}{2}\,|\langle -1,-1,1\rangle| = \dfrac{25}{2}\sqrt{(-1)^2+(-1)^2+1^2} = \boxed{\dfrac{25\sqrt{3}}{2}}$

**Tangent planes to level surfaces:**

*FACT:* The gradient vector $\nabla F(x_0, y_0, z_0)$ is **orthogonal** to the level surface $F(x,y,z) = k$ at the point $(x_0, y_0, z_0)$.

So, the *tangent plane* to the surface $f(x,y,z) = k$ at the point $(x_0, y_0, z_0)$ has the equation:

Under assumption $\nabla F(x_0,y_0,z_0) \neq 0$

$\nabla F(x_0,y_0,z_0)$ is normal to the tangent plane

$$\Longleftrightarrow \quad F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$

The normal line to the surface at the point $(x_0, y_0, z_0)$ is the line passing through $(x_0, y_0, z_0)$ and perpendicular to the tangent plane. Therefore its direction is given by the gradient vector

Explanation: WLOG assume that $F_z(x_0,y_0,z_0) \neq 0 \Longrightarrow$ near $(x_0,y_0,z_0)$ the level surface $F(x,y,z) = k$ is the graph of $z = f(x,y)$ for some differentiable function $f \Rightarrow$ the tangent plane to this graph at $(x_0,y_0,z_0)$ is

$$z - z_0 = \underbrace{f_x(x_0,y_0)}_{= -\frac{F_x(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}}^{\frac{\partial z}{\partial x}}(x-x_0) + \underbrace{f_y(x_0,y_0)}_{= -\frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}}^{\frac{\partial z}{\partial y}}(y-y_0)$$

EXAMPLE 12. *Find the equation of the tangent plane and normal line at the point $(1,0,5)$ to the surface $xe^{yz} = 1$.* $\Rightarrow F(x,y,z) = xe^{yz}$

The normal vector to the tangent plane is $\nabla F(1,0,5)$

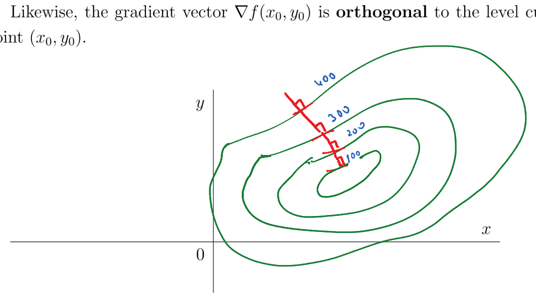
$F_x = e^{yz} \Rightarrow F_x(1,0,5) = e^0 = 1$, $F_z = xye^{yz} \Rightarrow F_z(1,0,5) = 0$

$F_y = xze^{yz} \Rightarrow F_y(1,0,5) = 1\cdot 5e^0 = 5 \Rightarrow$ The eq. of the tangent plane: $1\cdot(x-1) + 5\cdot(y-0) + 0\cdot(z-5) = 0 \Leftrightarrow$

$x - 1 + 5y = 0 \Leftrightarrow \boxed{x + 5y = 1}$. Normal line $\begin{cases} x = 1+t \\ y = 5t \\ z = 5 \end{cases}$



Likewise, the gradient vector $\nabla f(x_0, y_0)$ is **orthogonal** to the level curve $f(x,y) = k$ at the point $(x_0, y_0)$.




Consider a topographical map of a hill and let $f(x,y)$ represent the height above sea level at a point with coordinates $(x,y)$. Draw a curve of steepest ascent.