F19_LN_14_6

Monday, September 30, 2019 7:09 PM

F19_LN_1...

14.6: Directional Derivatives and the Gradient Vector

Recall that the two partial derivatives $f_x(x, y)$ and $f_y(x, y)$ of $f(x, y)$ represent the rate of change of f as we vary x (holding y fixed) and as we vary y (holding x fixed) respectively. In other words, $f_x(x, y)$ and $f_y(x, y)$ represent the rate of change of f in the directions of the unit vectors **i** and **j** respectively. Let's consider how to find the rate of change of f if we allow both x and y to change simultaneously, or how to find the rate of change of f in the direction of an arbitrary vector **u**.

$\hat{u}=(a,b)$

x_0, y_0

DEFINITION 1. *The rate of change of $f(x, y)$ in the direction of the unit vector $\hat{\mathbf{u}} = \langle a, b\rangle$ is called the* **directional derivative** *and it is denoted by $D_{\mathbf{u}}f(x, y)$.*

The **directional derivative** *of f at (x_0, y_0) in the direction of the unit vector $\hat{\mathbf{u}} = \langle a, b\rangle$ is*

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = \frac{d}{dh} f(x_0+ha, x_0+hb)\Big|_{h=0}$$

if this limit exists.

REMARK 2. By comparing the last definition with the definitions of the partial derivatives:

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \quad f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

we see that $a=1,\ b=0$, $\hat{u}=\langle 1,0\rangle=\hat{i}$ $\qquad$ $a=0$, $b=1$, $\hat{u}=\langle 0,1\rangle=\hat{j}$

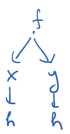
$$f_x(x_0, y_0) = D_{\hat{i}} f(x_0,y_0) \qquad \text{and} \qquad f_y(x_0, y_0) = D_{\hat{j}} f(x_0,y_0)$$

For computational purposes use the following theorem.

THEOREM 3. *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any* <u>*unit*</u> *vector $\hat{\mathbf{u}} = \langle a, b\rangle$ and*

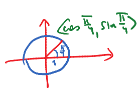
$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b. = \langle f_x(x,y), f_y(x,y)\rangle\cdot\langle a,b\rangle$$

Proof



$$D_u f(x_0,y_0) = \frac{d}{dh} f(\underbrace{x_0+ah}_{x(h)}, \underbrace{y_0+bh}_{y(h)})\Big|_{h=0} = \frac{\partial f}{\partial x}(x_0,y_0)\underbrace{\frac{\partial x}{\partial h}}_{a} + \frac{\partial f}{\partial y}(x_0,y_0)\cdot\underbrace{\frac{\partial y}{\partial h}}_{b}$$

EXAMPLE 4. *Find the rate of change $f(x, y) = x^3 + \sin(xy)$ at the point $(1, \pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.*



$$\hat{u} = \langle \cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}\rangle = \langle \underbrace{\tfrac{1}{\sqrt{2}}}_{a}, \underbrace{\tfrac{1}{\sqrt{2}}}_{b}\rangle$$

$$f_x = 3x^2 + y\cos(xy) \Rightarrow f_x(1,\tfrac{\pi}{2}) = 3\cdot 1^2 + \tfrac{\pi}{2}\underbrace{\cos\tfrac{\pi}{2}}_{0} = 3$$

$$f_y = x\cos(xy) \Rightarrow f_y(1,\tfrac{\pi}{2}) = 1\cdot\cos\tfrac{\pi}{2} = 0$$

$$D_{\hat{u}} f(1,\tfrac{\pi}{2}) = \underbrace{f_x(1,\tfrac{\pi}{2})}_{3}\cdot\tfrac{1}{\sqrt{2}} + \underbrace{f_y(1,\tfrac{\pi}{2})}_{0}\cdot\tfrac{1}{\sqrt{2}} = \boxed{\tfrac{3}{\sqrt{2}}}$$

The Directional Derivative As The Dot Product Of Two Vectors. Gradient.

DEFINITION 5. *The* **gradient** *of $f(x, y)$ is the vector function ∇f defined by*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Notations for gradient: **grad**f or ∇f which is read "del f". nabla

EXAMPLE 6. *Find the gradient of $f = \cos(xy) + e^x$ at (0,3).*

$f_x = -y\sin(xy) + e^x \Rightarrow f_x(0,3) = -3\cdot\underbrace{\sin 0}_{0} + \underbrace{e^0}_{1} = 1$

$f_y = -x\sin xy \Rightarrow f_y(0,3) = 0$

$\Rightarrow \nabla f(0,3) = \langle 1, 0\rangle = \hat{i}$

By Theorem 3 we have:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y)\cdot\langle a,b\rangle = \nabla f(x,y)\cdot\hat{u}$$

Formula for the directional derivative using the gradient vector:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \underbrace{\hat{\mathbf{u}}}_{\text{unit}}.$$

EXAMPLE 7. *Find the directional derivative for f from Example 6 at $(0,3)$ in the direction* $\vec{u} = \langle 3,4\rangle$.

The unit vector $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$, $|\vec{u}| = \sqrt{3^2+4^2} = \sqrt{25} = 5 \Rightarrow$

$\hat{u} = \frac{\vec{u}}{5} = \langle \frac{3}{5}, \frac{4}{5}\rangle$

$\nabla f(0,3) = \langle 1, 0\rangle$

$\Rightarrow D_{\vec{u}}f(0,3) = \langle 1,0\rangle\cdot\langle\frac{3}{5},\frac{4}{5}\rangle = \frac{3}{5}$

The directional derivative of function of *three* variables

THEOREM 8. *If f is a differentiable function of x, y and z, then f has a directional derivative in the direction of any unit vector $\hat{\mathbf{u}} = \langle a, b, c\rangle$ and*

$$D_{\mathbf{u}}f(x,y) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \nabla f\cdot\hat{\mathbf{u}},$$

where

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is the gradient vector of $f(x,y,z)$.

EXAMPLE 9. *Find the directional derivative of* $f(x, y, z) = z^3 - x^2 y$ *at the point* $(1, 6, 2)$ *in the direction* $\mathbf{u} = \langle 1, -2, 3\rangle$.

$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$ $\quad |\vec{u}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} \Rightarrow \hat{u} = \frac{1}{\sqrt{14}} \langle 1, -2, 3\rangle$

$f_x = -2xy \Rightarrow f_x(1,6,2) = -2 \cdot 1 \cdot 6 = -12$

$f_y = -x^2 \Rightarrow f_y(1,6,2) = -1$

$f_z = 3z^2 \Rightarrow f_z(1,6,2) = 3 \cdot 2^2 = 12$

$\Rightarrow \nabla f(1,6,2) = \langle -12, -1, 12\rangle$

$D_u f = \nabla f(1,6,2) \cdot \hat{u} = \langle -12, -1, 12\rangle \cdot \frac{1}{\sqrt{14}} \langle 1, -2, 3\rangle =$

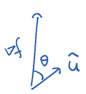
$= \frac{1}{\sqrt{14}} \left((-12) \cdot 1 + (-1) \cdot (-2) + 12 \cdot 3\right) = \boxed{\frac{26}{\sqrt{14}}}$

QUESTION: In which of all possible directions does f *change fastest and what is the maximum rate of change.*

ANSWER is provided by the following theorem:

THEOREM 10. *Suppose* f *is a differentiable function of two or three variables. The maximum value of the directional derivative* $D_{\mathbf{u}} f$ *is* $|\nabla f|$ *and it occurs when* $\mathbf{u}$ *has the same direction as the gradient vector* ∇f.

Proof. $D_u f = \nabla f \cdot \hat{u} = |\nabla f| \underbrace{|\hat{u}|}_{1} \underbrace{\cos\theta}_{\leq 1} \leq |\nabla f|$



$0 \leq \theta \leq \pi$

When the equality holds? $\Leftrightarrow \cos\theta = 1 \Leftrightarrow \theta = 0 \Leftrightarrow$

$\hat{u}$ is in the direction of ∇f. ∎

EXAMPLE 11. *Suppose that the temperature at a point* (x, y, z) *in the space is given by*

$$T(x, y, z) = \frac{100}{1 + x^2 + y^2 + z^2}, \quad = 100\left(1+x^2+y^2+z^2\right)^{-1}$$

where T *is measured in* $^\circ C$ *and* x, y, z *in meters.*

(a) *In which direction does the temperature increase fastest at the point* $(1, 1, -1)$*?*

It is in the direction of $\nabla T(1,1,-1)$

$$\frac{\partial T}{\partial x} = -100\left(1+x^2+y^2+z^2\right)^{-2}\cdot 2x = -\frac{200x}{(1+x^2+y^2+z^2)^2} \Rightarrow \frac{\partial T}{\partial x}(1,1,-1) = -\frac{200}{4^2} = -\frac{25}{2}$$

$$\frac{\partial T}{\partial y} = -\frac{200y}{(1+x^2+y^2+z^2)^2} \Rightarrow \frac{\partial T}{\partial y}(1,1,-1) = -\frac{25}{2} \rightarrow$$

$$\frac{\partial T}{\partial z} = -\frac{200z}{(1+x^2+y^2+z^2)^2} \Rightarrow \frac{\partial T}{\partial z}(1,1,-1) = \frac{25}{2} \rightarrow \nabla T(1,1,-1) = \left\langle -\frac{25}{2}, -\frac{25}{2}, \frac{25}{2}\right\rangle = \frac{25}{2}\langle -1,-1,1\rangle$$

Answer: In the direction of $\langle -1,-1,1\rangle$ (or $\frac{25}{2}\langle -1,-1,1\rangle$)

(b) *What is the maximum rate of increase?*

It is $|\nabla T(-1,-1,1)| = \frac{25}{2}|\langle -1,-1,1\rangle| = \frac{25}{2}\sqrt{(-1)^2+(-1)^2+1^2} = \boxed{\frac{25\sqrt{3}}{2}}$

Tangent planes to level surfaces:

FACT: The gradient vector $\nabla F(x_0, y_0, z_0)$ is **orthogonal** to the level surface $F(x, y, z) = k$ at the point (x_0, y_0, z_0).

So, the *tangent plane* to the surface $f(x, y, z) = k$ at the point (x_0, y_0, z_0) has the equation:

Under assumption $\nabla F(x_0,y_0,z_0) \neq 0$:

$$\Leftrightarrow F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

$\nabla F(x_0,y_0,z_0)$ is normal to the level surface

The normal line to the surface at the point (x_0, y_0, z_0) is the line passing through (x_0, y_0, z_0) and perpendicular to the tangent plane. Therefore its direction is given by the gradient vector

EXAMPLE 12. *Find the equation of the tangent plane and normal line at the point* $(1, 0, 5)$ *to the surface* $xe^{yz} = 1$. $\Rightarrow F(x,y,z) = xe^{yz}$

$F_x = e^{yz} \Rightarrow F_x(1,0,5) = e^0 = 1$

$F_y = xze^{yz} \Rightarrow F_y(1,0,5) = 1\cdot 5\cdot e^0 = 5$

$F_z = xye^{yz} \Rightarrow F_z(1,0,5) = 0$

The eq. of the tangent plane is $1(x-1) + 5(y-0) + 0(z-5) = 0 \Leftrightarrow x - 1 + 5y = 0 \Leftrightarrow \boxed{x+5y=1}$

Normal line $\parallel \langle 1,5,0\rangle$ through $(1,0,5)$

$\boxed{x = 1+t,\ y = 5t,\ z = 5}$

Explanation: WLOG assume that $F_z(x_0,y_0,z_0) \neq 0 \Rightarrow$ near (x_0,y_0,z_0) the level surface $F(x,y,z)=k$ is a graph of some differentiable function $z=f(x,y)$. The tangent plane to this graph at (x_0,y_0,z_0) is

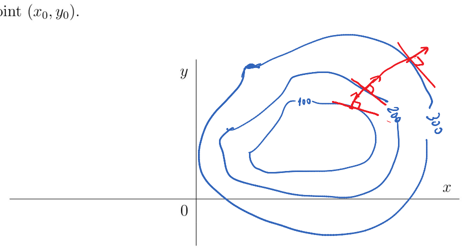
$$z - z_0 = \underbrace{f_x(x_0,y_0)}_{\frac{\partial z}{\partial x}}(x-x_0) + \underbrace{f_y(x_0,y_0)}_{\frac{\partial z}{\partial y}}(y-y_0)$$

By implicit differentiation $f_x(x_0,y_0) = -\frac{F_x(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}$, $f_y(x_0,y_0) = -\frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}$

$\Downarrow$ (multiplying by $F_z(x_0,y_0,z_0)$ and moving all to the left-hand side)

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

Likewise, the gradient vector $\nabla f(x_0, y_0)$ is **orthogonal** to the level curve $f(x, y) = k$ at the point (x_0, y_0).



Consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y). Draw a curve of steepest ascent.