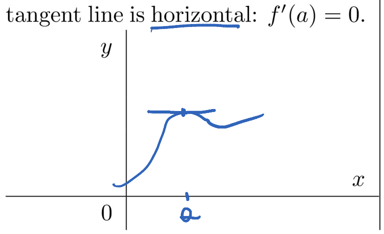
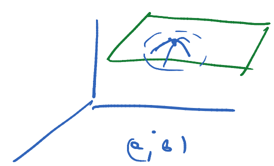
F19_LN_14_7

Tuesday, October 8, 2019 9:28 PM

F19_LN_1...

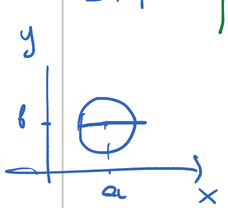
14.7: Maximum and minimum values

Function $y = f(x)$	Function of two variables $z = f(x, y)$
DEFINITION 1. *A function $f(x)$ has a local maximum at $x = a$ if $f(a) \geq f(x)$ when x is near a (i.e. in a neighborhood of a). A function f has a local minimum at $x = a$ if $f(a) \leq f(x)$ when x is near a.*	or relative DEFINITION 2. *A function $f(x, y)$ has a local maximum at $(x, y) = (a, b)$ if $f(a, b) \geq f(x, y)$ when (x, y) is near (a, b) (i.e. in a neighborhood of (a, b)). A function f has a local minimum at $(x, y) = (a, b)$ if $f(a, b) \leq f(x, y)$ when (x, y) is near (a, b).* (a, b)
.	
If the inequalities in this definition hold for ALL points x in the domain of f, then f has an **absolute max** (or **absolute min**) at a	If the inequalities in this definition hold for ALL points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).
If the graph of f has a tangent line at a local extremum, then the tangent line is horizontal: $f'(a) = 0$.	If the graph of f has a tangent plane at a local extremum, then the tangent PLANE is horizontal.
	

THEOREM 3. *If f has a local extremum (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives exist there, then*

$$f_x(a, b) = f_y(a, b) = 0 \quad (or,\ equivalently, \nabla f(a, b) = 0.) \Rightarrow D_u f(a,b) = 0$$

Explanation



If (a,b) is a local maximum of f $\Rightarrow$ $x = a$ is local max. of function $x \mapsto \underbrace{f(x, b)}_{\text{a function of single variable } x} \overset{\text{Calc 1}}{\Longrightarrow} \frac{\partial}{\partial x} f(a,b) = 0$

Similarly $y = b$ is a loc. max. of $y \mapsto f(a,y) \overset{\text{Calc 1}}{\Longrightarrow}$ $\frac{\partial f}{\partial y}(a,b) = 0$

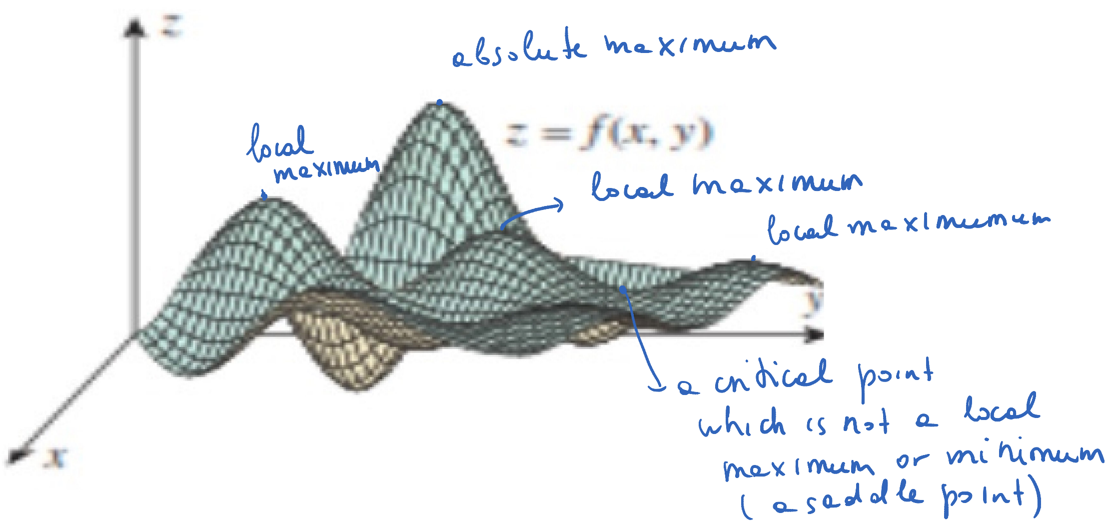
DEFINITION 4. *A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of this partial derivatives does not exist, is called a* **critical point** *of f.*

At a critical point, a function could have a local max or a local min, or neither.

derivatives does not exist, is called a **critical point** *of f.*

At a critical point, a function could have a local max or a local min, or neither. We will be concerned with two important questions:

- Are there any local or absolute extrema?
- If so, where are they located?

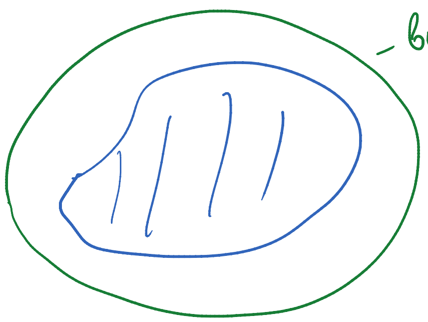


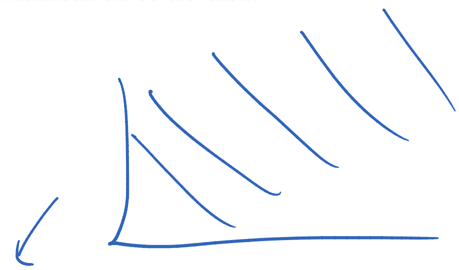
https://www.slideshare.net/abdulazizuinmlg/multivariate-calculus-mhsw-2

in $\mathbb{R}$	in $\mathbb{R}^2$
closed interval $[a, b]$ $\{x \in \mathbb{R} : a \le x \le b\}$ a b	closed set → domains with boundaries Examples D_1 D_2 -2 2 -1 $D_1 = \{(x,y) : x^2+y^2 \le 1\}$ $D_2 = \{(x,y) : -2 \le x \le 2, -1 \le y \le 1\}$
open interval (a, b) a b	open set strict inequalities. any point belongs to a set together with a small disk
end points of an interval $x = a$, $x = b$	boundary points ∂D the boundary of D ∂D_1 ∂D_2

$x = a$, $x = b$	the boundary of D  ∂D_1 ∂D_2

DEFINITION 5. *A* **bounded set** *in* $\mathbb{R}^2$ *is one that contained in some disk.*

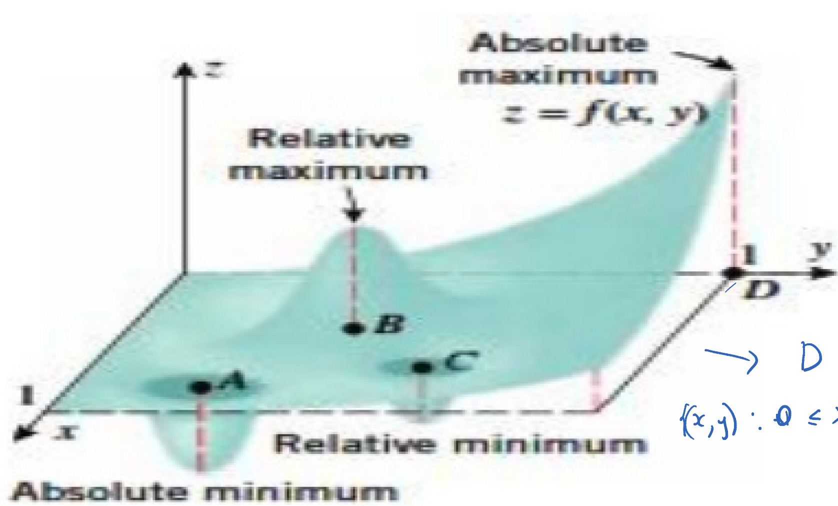




first quadrant is unbounded

THE EXTREME VALUE THEOREM:

Function $y = f(x)$	Function of two variables $z = f(x, y)$
If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(x_1)$ and an absolute minimum value $f(x_2)$ at some points x_1 and x_2 in $[a, b]$.	If f is continuous on a closed bounded set $\mathcal{D}$ in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in $\mathcal{D}$.



$\to$ D is a square $\{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$

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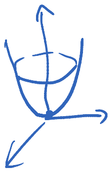
EXAMPLE 6. *Find extreme values of* $f(x, y) = x^2 + y^2$. $(x,y) \in \mathbb{R}^2$

	Local	Absolute

$\lim f(x,y) = \infty$ $f_x = 2x = 0$

EXAMPLE 6. *Find extreme values of* $f(x,y)=x^2+y^2$. $(x,y)\in\mathbb{R}$



	Local	Absolute
Maximum	none	none $\to \lim_{(x,y)\to\infty} f(x,y)=\infty$
Minimum	at $(0,0)$	at $(0,0)$

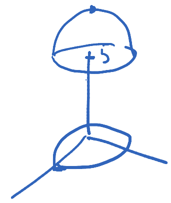
Domain:

$f_x = 2x = 0$
$f_y = 2y = 0$ $\Rightarrow$ $(x,y)=(0,0)$ is the only critical point and $f(0,0)=0\le x^2+y^2=f(x,y)$ $\Rightarrow$ $(0,0)$ is absolute minimum

EXAMPLE 7. *Find extreme values of* $f(x,y)=5+\sqrt{1-x^2-y^2}$. $D=\{(x,y):\ x^2+y^2\le 1\}$

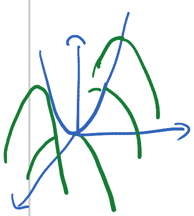
	Local	Absolute
Maximum	at $(0,0)$	at $(0,0)$
Minimum	none inside of D	on ∂D

Domain: $\{(x,y):\ x^2+y^2\le 1\}$



the graph is hyperbolic paraboloid

EXAMPLE 8. *Find extreme values of* $f(x,y)=x^2-y^2$. $(x,y)\in\mathbb{R}^2$



	Local	Absolute
Maximum	none	none
Minimum	none	none

Domain:

$f_x = 2x = 0$
$f_y = -2y = 0$ $\Rightarrow$ the only critical point is $(0,0)$

$x \mapsto f(x,0)=x^2$ has local min at $x=0$
$y \to f(0,y)=-y^2$ has local max at $y=0$

REMARK 9. Example 8 illustrates so called **saddle point** of f. Note that the graph of f crosses its tangent plane at (a,b).

ABSOLUTE MAXIMUM AND MINIMUM VALUES on a closed bounded set.

THE EXTREME VALUE THEOREM:

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

1. Find the values of f at the critical points of f in (a,b).

2.Find the values of f at the end-points of the interval.

3.The largest of the values from steps 1&2 is the absolute max value; the smallest of the values from steps 1&2 is the absolute min value.

To find the absolute max and min values of a continuous function f on a closed bounded set D:

1. Find the values of f at the critical points of f in inside D.

$f_x=0,\ f_y=0$ inside D

2.Find the extreme values of f on the boundary of D.(This usually involves either the Calculus I approach or the Lagnrage multiplies nethod of section 14.8 for this work.)

3. The largest of the values from steps 1&2 is the absolute maximum value; the smallest of the values from steps 1&2 is the absolute minimum value.

steps 1&2 is the absolute max value; the smallest of the values from steps 1&2 is the absolute min value.	maximum value; the smallest of the values from steps 1&2 is the absolute minimum value.

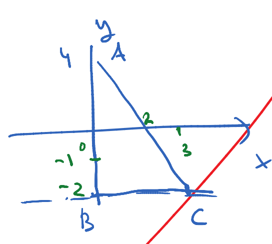
- The quantity to me maximized/minimized is expressed in terms of variables (as few as possible!)
- Any constraints that are presented in the problem are used to reduce the number of variables to the point they are independent,
- After computing partial derivatives and setting them equal to zero you get purely algebraic problem (but it may be hard.)
- Sort out extreme values to answer the original question.

EXAMPLE 10. *A lamina occupies the region* $D = \{(x,y):\ 0 \le x \le 3,\ \ -2 \le y \le 4-2x\}$. *The temperature at each point of the lamina is given by*

$$T(x,y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.$$

Find the hottest and coldest points of the lamina.

1. Draw D – triangle ABC



2. Find critical points inside D

$T_x = 4(2x + y - 3) = 0$

$T_y = 4(x + 4y + 2) = 0$ $\Longleftrightarrow$

$\begin{cases} 2x + y = 3 \to Eq1 \\ x + 4y = -2 \ \times 2 \ Eq2 \end{cases}$ Eliminate x

$2x + 8y = -4 \to Eq2'$

$(Eq1) - (Eq2')$: $-7y = 7 \Rightarrow y = -1$

Plug into $(Eq2)$: $x - 4 = -2 \Rightarrow x = 2$

Is $(2,-1)$ inside of D? $-2 \le -1 \le \underbrace{4 - 2\cdot 2}_{0} \Rightarrow$ Yes

Value $T(2,-1)$? $T(2,-1) = 4(\underbrace{4 - \cancel{2} + \cancel{2} - 6 - 2}_{-4}) + 10 = -16 + 10 = -6$

3. Describe the boundary: $\partial D = \overline{AB} \cup \overline{AC} \cup \overline{BC}$

Value $T(2,-1)$: $T(2,-1) = 4(\underbrace{4-2+2-6}_{-4}\ -2)+10 = -16+10 = \dots$

3. Describe the boundary: $\partial D = \overline{AB} \cup \overline{AC} \cup \overline{BC}$

4. Calculate T on the vertices: $T(A) = T(0,4) = 4(\underbrace{32+8}_{40})+10 = \underline{170}$

$T(B) = T(0,-2) = 4(8-4)+10 = \underline{26}$; $T(C) = T(3,-2) = \dots = \underline{2}$

5. Find critical point on the edges (by parametrizing each edge)

$T(x,y) = 4(x^2+xy+2y^2-3x+2y)+10$

$\overline{AB}$	$\overline{BC}$	$\overline{AC} = \{(x,y): y = 4-2x,\ 0 \le x \le 3\}$
$x=0,\ y=t,\ -2 \le t \le 4$ Plug into T: $\underbrace{T(0,t)}_{T_1(t)} = 4(2t^2+2t)+10$	$x=t,\ y=-2,\ 0 \le t \le 3$ Plug into T: $T(t,-2) = 4(t^2-2t+8-3t-4)+10 =$ $= \underbrace{4(t^2-5t+4)+10}_{T_2(t)}$	$x=t,\ y=4-2t,\ 0 \le t \le 3$ Plug into T: $T(t, 4-2t) =$ $= 4(t^2+t(4-2t)+2(4-2t)^2-3t+2(4-2t))+10 = \dots =$ $= 4(7t^2-35t+40)+10$
Solve $\frac{d}{dt}T_1(t)=0$ in $(-2,4)$	$\frac{d}{dt}T_2(t)=0$ in $(0,3)$	$\frac{d}{dt}T_3(t)=0$ in $(0,3)$
$\frac{dT_1}{dt} = 4(4t+2)=0 \Rightarrow t=-\frac{1}{2}$ and it is inside $(-2,4)$	$\frac{dT_2}{dt}(t) = 4(2t-5)=0$ $t=\frac{5}{2} \in (0,3)$	$\frac{dT_3}{dt}(t) = 4(14t-35)=0$ $t=\frac{35}{14}=\frac{5}{2} \in (0,3)$
Find the values		
$\underset{T(0,-\frac{1}{2})}{T_1(-\tfrac{1}{2})} = 4(\underbrace{2\cdot\tfrac{1}{4}}_{\frac{1}{2}} + \underbrace{2\cdot(-\tfrac{1}{2})}_{-1})+10 = -2+10 = \underline{8}$	$\underset{T(\frac{5}{2},-2)}{T_2(\tfrac{5}{2})} = 4(\underbrace{\tfrac{25}{4}-5\cdot\tfrac{5}{2}}_{-\frac{25}{4}}+4)+10 = -25+16+10 = \underline{1}$	$\underset{T(\frac{5}{2},\,\underbrace{4-2\cdot\frac{5}{2}}_{-1})}{T_3(\tfrac{5}{2})} = 4\cdot(7\cdot\tfrac{25}{4}-35\cdot\tfrac{5}{2}+40)+10 = \dots = \underline{-5}$

Comparing all 7 values we get $T_{max} = 170$ at $A=(0,4)$, $T_{min} = -6$ at $(2,-1)$

Local/Relative Extrema

Second derivatives test:

Suppose f'' is continuous near a and $f'(c)=0$ (i.e. a is a critical point).	Suppose that the second partial derivatives of f are continuous near (a,b) and $\nabla f(a,b) = \mathbf{0}$ (i.e. (a,b) is a critical point). Let $\mathcal{D} = \mathcal{D}(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$
• If $f''(c) > 0$ then $f(c)$ is a local minimum.	• If $\mathcal{D} > 0$ and $f_{xx}(a,b) > 0$ then $f(a,b)$ is a local minimum. (or $f_{yy}(a,b) > 0$)
• If $f''(c) < 0$ then $f(c)$ is a local maximum.	• If $\mathcal{D} > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum. (or $f_{yy}(a,b) < 0$)
	• If $\mathcal{D} < 0$ then $f(a,b)$ is not a local extremum (saddle point).

NOTE:

• If $f''(c)=0$, then the test gives no information.	If $\mathcal{D}=0$ or does not exist, then the test gives no information. fails. To remember formula for $\mathcal{D}$:

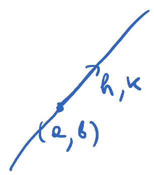
fails.

To remember formula for $\mathcal{D}$:

$$\mathcal{D} = f_{xx}f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Hessian

Sketch of the proof of the Second Derivative test



$g(t) = f(a + th, b + tk)$ $g(0) = f(a, b)$

$g'(0) = 0 \quad (\Leftrightarrow D_{(h,k)} f(a, b) = 0)$

$g''(0) = f_{xx}(a, b) h^2 + 2 f_{xy}(a, b) hk + f_{yy}(a, b) k^2 \ \substack{\geq 0 \\ \leq 0}$ for any (h, k)

$\Downarrow$

reduce to analyzing if a quadr. poly.

$f_{xx} s^2 + 2 f_{xy} s + f_{yy}$ has real root

$D = -\frac{1}{4}$ discriminant of this polynomial

EXAMPLE 11. *Use the Second Derivative Test to confirm that a local cold point of the lamina in the previous Example is* $(2, -1)$. $T(x, y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10$

$T_x = 4(2x + y - 3) = 0$ See prev. Ex $\Rightarrow$ $(x, y) = (2, -1)$ is the only critical point

$T_y = 4(x + 4y + 2) = 0$

Check the second derivative test for $(2, -1)$

$T_{xx} = 8$, $T_{xy} = 4$

$T_{yy} = 16$

$D = \begin{vmatrix} 8 & 4 \\ 4 & 16 \end{vmatrix} = 8 \cdot 16 - 16 = 7 \cdot 16 > 0$

$T_{xx}(2, -1) = 8 > 0 \rightarrow (2, -1)$ is a local minimum

EXAMPLE 12. *Find the local maximum and minimum values and saddle point(s) of the function* $f(x, y) = 4xy - x^4 - y^4$.

1. Find critical points

$f_x = 4y - 4x^3 = 0 \Rightarrow y = x^3 \Rightarrow y = (y^3)^3 = y^9$

$f_y = 4x - 4y^3 = 0 \Rightarrow x = y^3$ $x = y^3$

$\underbrace{y^9 - y}_{\text{factor it}} = 0 \Rightarrow y(\underbrace{y^8}_{(y^4)^2} - 1) = 0 \Rightarrow$

$a^2 - 1 = (a-1)(a+1)$

$\Leftrightarrow y(\overset{(y^2)^2}{y^4} - 1)(y^4 + 1) = 0 \Leftrightarrow y(y^2 - 1)(y^2 + 1)(y^4 + 1) = 0$

$\Leftrightarrow y(y - 1)(y + 1)\underbrace{(y^2 + 1)}_{>0}\underbrace{(y^4 + 1)}_{>0} = 0 \Rightarrow$

$\Leftrightarrow y(y-1)(y+1)\underbrace{(y^2+1)}_{>0}\underbrace{(y^4+1)}_{>0}=0 \Rightarrow$

$\Rightarrow \quad y=0$ or $y=1$ or $y=-1$

$\Downarrow \qquad \Downarrow \; x=y^3$

$x=0 \qquad x=1 \qquad x=-1$

$\Rightarrow$ There are 3 critical points $(0,0)$, $(1,1)$, and $(-1,-1)$

2. Find second partial derivatives and check second derivatives test at every crit. point

$f_{xx} = -12x^2$

$f_{xy} = 4$

$f_{yy} = -12y^2$

Apply the test

	$(0,0)$	$(1,1)$	$(-1,-1)$
$f_{xx} = -12x^2$	0	$-12<0$	$-12<0$
$f_{xy} = 4$	4	4	4
$f_{yy} = -12y^2$	0	-12	-12
$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$	$\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$ saddle point	$\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144-16 > 0$ local maximum	$\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} > 0$ local maximum