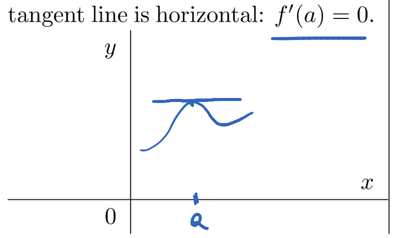
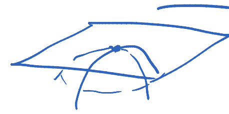
F19_LN_14_7

Tuesday, October 8, 2019 9:27 PM

F19_LN_1...

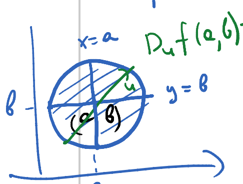
14.7: Maximum and minimum values

Function $y = f(x)$	Function of two variables $z = f(x, y)$
DEFINITION 1. *A function $f(x)$ has a local maximum at $x = a$ if $f(a) \geq f(x)$ when x is near a (i.e. in a neighborhood of a). A function f has a local minimum at $x = a$ if $f(a) \leq f(x)$ when x is near a.* .	or relative DEFINITION 2. *A function $f(x, y)$ has a local maximum at $(x, y) = (a, b)$ if $f(a, b) \geq f(x, y)$ when (x, y) is near (a, b) (i.e. in a neighborhood of (a, b)). A function f has a local minimum at $(x, y) = (a, b)$ if $f(a, b) \leq f(x, y)$ when (x, y) is near (a, b).*
If the inequalities in this definition hold for ALL points x in the domain of f, then f has an **absolute max** (or **absolute min**) at a	If the inequalities in this definition hold for ALL points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).
If the graph of f has a tangent line at a local extremum, then the tangent line is horizontal: $f'(a) = 0$.	If the graph of f has a tangent plane at a local extremum, then the tangent PLANE is horizontal.
	

THEOREM 3. *If f has a local extremum (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives exist there, then*

$$f_x(a, b) = f_y(a, b) = 0 \quad (\textit{or, equivalently}, \nabla f(a, b) = 0.) \iff D_u f(a,b)=0$$

Explanation: If f has a local max. at (a,b) then the function $x \mapsto f(x,b)$ has a local max at $x=a$ $\overset{\text{Calc 1}}{\Longrightarrow}$ $\frac{\partial f}{\partial x}(a,b)=0$



Similarly, the function $y \mapsto f(a,y)$ has a local max. at $y=b$ $\overset{\text{Calc 1}}{\Longrightarrow}$ $\frac{\partial}{\partial y} f(a,b) = 0$

DEFINITION 4. *A point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or one of this partial derivatives does not exist, is called a* **critical point** *of f.*

At a critical point, a function could have a local max or a local min, or neither.

DEFINITION 1. *A point (a,b) such that $f_x(a,b) = 0$ and $f_y(a,b) = 0$, or one of this partial derivatives does not exist, is called a* **critical point** *of f.*

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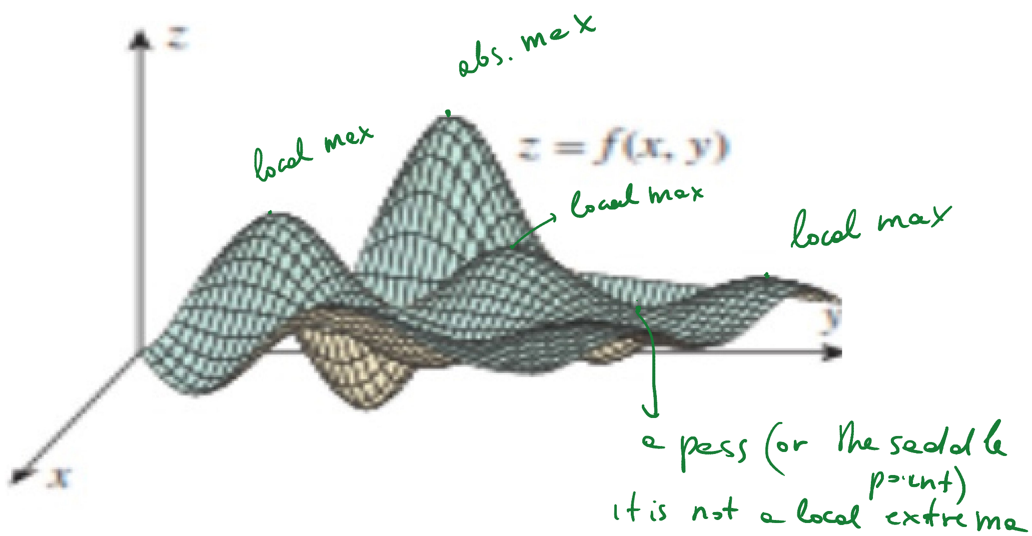
We will be concerned with two important questions:

- Are there any local or absolute extrema?
- If so, where are they located?

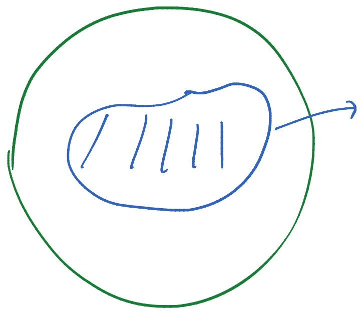


https://www.slideshare.net/abdulazizuinmlg/multivariate-calculus-mhsw-2

in $\mathbb{R}$	in $\mathbb{R}^2$
closed interval $[a,b]$ a b Igor Zelenko at 10/9/2019 9:38 AM $\{x \in \mathbb{R} : a \le x \le b\}$	closed set – containes boundary D_1 D_2 $D_1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$ $D_2 = \{(x,y) \in \mathbb{R}^2 : -2 \le x \le 2, -1 \le y \le 1\}$
open interval (a,b) a b	open set strict ineq. without boundaries $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ $\{(x,y) \in \mathbb{R}^2 : -2 < x < 2, -1 < y < 1\}$
end points of an interval	boundary points Notation : ∂D is the boundary of D

	$\{(x,y)\in\mathbb{R}: x^2+y^2<1\}$ $\{(x,y)\in\mathbb{R}: -1<y<1\}$
end points of an interval $x=a$, $x=b$	boundary points Notation: ∂D is the boundary of D $\partial D_1 = \{(x,y): x^2+y^2=1\}$

DEFINITION 5. *A* **bounded set** *in* $\mathbb{R}^2$ *is one that contained in some disk.*



bounded

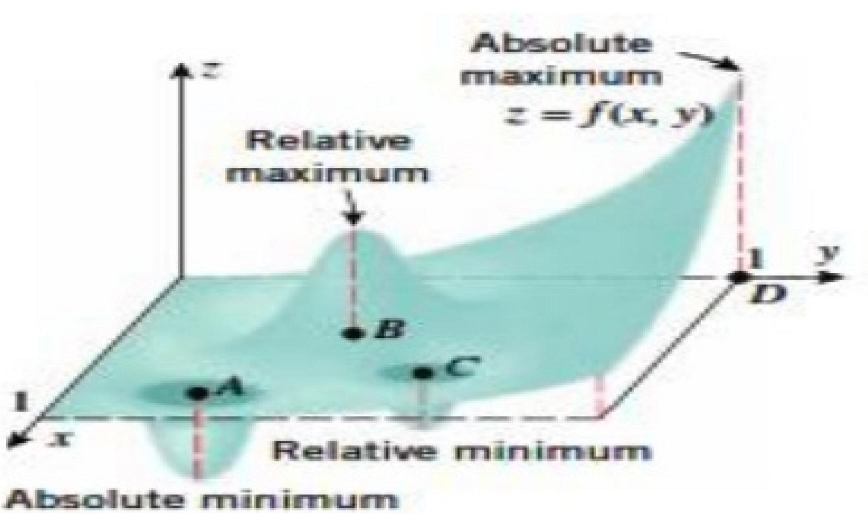
1st quadrant

not bounded

THE EXTREME VALUE THEOREM:

compact

Function $y=f(x)$	Function of two variables $z=f(x,y)$
If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(x_1)$ and an absolute minimum value $f(x_2)$ at some points x_1 and x_2 in $[a,b]$.	If f is continuous on a closed bounded set $\mathcal{D}$ in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points (x_1,y_1) and (x_2,y_2) in $\mathcal{D}$.



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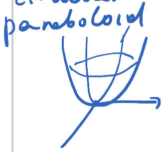
EXAMPLE 6. *Find extreme values of* $f(x, y) = x^2 + y^2$. $(x$

circular paraboloid



	Local	Absolute
Maximum	none	none $\rightarrow f(x,y)\to\infty$ as $(x,y)\to\infty$
Minimum	at $(0,0)$	at $(0,0)$

Domain:

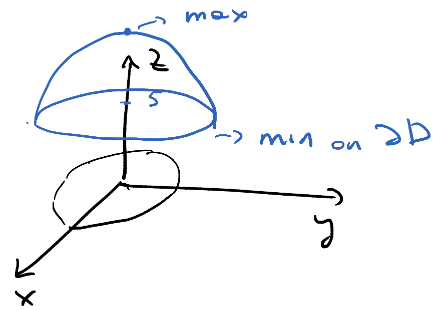
$f_x = 2x = 0$
$f_y = 2y = 0$ $\Rightarrow$ $(0,0)$ is the only critical point

$f(0,0) = 0 \leq x^2+y^2 = f(x,y)$ for every (x,y) $\Rightarrow$ $(0,0)$ is absolute maximum

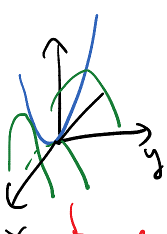
EXAMPLE 7. *Find extreme values of* $f(x, y) = 5 + \sqrt{1 - x^2 - y^2}$. $D = \{(x,y) : x^2+y^2 \leq 1\}$

	Local	Absolute
Maximum	at $(0,0)$	at $(0,0)$
Minimum	not inside D	on the boundary ∂D

Domain: $D = \{(x,y) : x^2+y^2 \leq 1\}$



EXAMPLE 8. *Find extreme values of* $f(x, y) = x^2 - y^2$. $(x,y) \in \mathbb{R}^2$



	Local	Absolute
Maximum	none	none
Minimum	none	none

Domain:

$f_x = 2x = 0$
$f_y = 2y = 0$

$(0,0)$ is the only critical point, but it is neither local minimum nor local maximum

$x \mapsto f(x,0) = x^2$ has a local min at $x=0$
$y \mapsto f(0,y) = -y^2$ has a local max at $y=0$

$(0,0)$ is a saddle point

REMARK 9. Example 8 illustrates so called **saddle point** of f. Note that the graph of f crosses its tangent plane at (a, b).

ABSOLUTE MAXIMUM AND MINIMUM VALUES on a closed bounded set.

THE EXTREME VALUE THEOREM:

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:	*To find the absolute max and min values of a continuous function f on a closed bounded set D:*
1. Find the values of f at the critical points of f in (a, b).	1. Find the values of f at the critical points of f in D. $f_x = 0$, $f_y = 0$ in D
2.Find the values of f at the end-points of the interval.	2.Find the extreme values of f on the boundary of D.(This usually involves either the Calculus I approach or the Lagnrage multiplies nethod of section 14.8 for this work.)

2.Find the values of f at the end points of the interval.	2.Find the extreme values of f on the boundary of D.(This usually involves either the Calculus I approach or the Lagnrage multiplies nethod of section 14.8 for this work.)
3.The largest of the values from steps 1&2 is the absolute max value; the smallest of the values from steps 1&2 is the absolute min value.	3. The largest of the values from steps 1&2 is the absolute maximum value; the smallest of the values from steps 1&2 is the absolute minimum value.

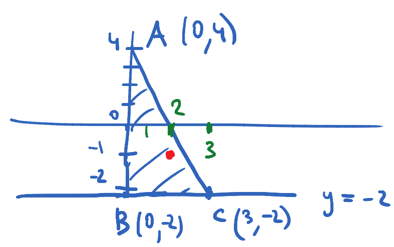
- The quantity to me maximized/minimized is expressed in terms of variables (as few as possible!)
- Any constraints that are presented in the problem are used to reduce the number of variables to the point they are independent,
- After computing partial derivatives and setting them equal to zero you get purely algebraic problem (but it may be hard.)
- Sort out extreme values to answer the original question.

EXAMPLE 10. *A lamina occupies the region* $D = \{(x,y):\ 0 \le x \le 3,\ \ -2 \le y \le 4-2x\}$. *The temperature at each point of the lamina is given by*

$$T(x,y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10.$$

Find the hottest and coldest points of the lamina.

1) Sketch the region
2) Find critical points inside D



$$\frac{\partial T}{\partial x} = 4(2x+y-3) = 0$$
$$\frac{\partial T}{\partial y} = 4(x+4y+2) = 0 \iff$$

$$\begin{cases} 2x+y = 3 & (Eq1) \\ x+4y=-2 & (Eq2)\ \times 2 \end{cases}$$
Eliminate x

$2x+8y=-4$ (Eq2')

$(Eq1)-(Eq2') = -7y = 7 \Rightarrow y=-1 \overset{Eq2}{\Rightarrow} x-4=-2 \Rightarrow x=2 \Rightarrow (2,-1)$ is a critical point

Is $(2,-1)$ in D Yes (from sketch). Analytically: $0 \le 2 \le 3$ $\quad -2 \le -1 \le 4-2\cdot 2$ ✓

$2x + 8y = -4$ (Eq 2')

$x - 4 = -2 \Rightarrow x = 2 \Rightarrow (2,-1)$ is a critical point

Is $(2,-1)$ in D? Yes (from sketch). Analytically: $0 \le 2 \le 3$, $-2 \le -1 \le \underbrace{4 - 2\cdot 2}_{0}$ ✓

$T(2,-1) = 4(\underbrace{4 - \cancel{2} + \cancel{2} - 6}_{-4} - 2) + 10 = -16 + 10 = \underline{-6}$

3) Describe the boundary $\partial D = \overline{AB} \cup \overline{BC} \cup \overline{AC}$

4) Calculate T at each vertex: $T(A) = T(0,4) = 4(32 + 8) + 10 = \underline{170}$

$T(x,y) = 4(x^2 + xy + 2y^2 - 3x + 2y) + 10$

$T(B) = T(0,-2) = 4(8 - 4) + 10 = \underline{26}$

$T(C) = T(3,-2) = 4(\cancel{9} - 6 + 8 - \cancel{9} - 4) + 10 = -8 + 10 = \underline{2}$

5) Find critical points inside each edge by parametrizing them

$\overline{AB}$	$\overline{BC}$	$\overline{AC} = \{(x,y) : y = 4 - 2x, 0 \le x \le 3\}$
$x = 0,\ y = t,\ -2 \le t \le 4$	$x = t,\ y = -2,\ 0 \le t \le 3$	$x = t,\ y = 4 - 2t,\ 0 \le t \le 3$
Plug in T: $\underbrace{T(0,t)}_{T_1(t)} = 4(2t^2 + 2t) + 10$	$\underbrace{T(t,-2)}_{T_2(t)} = 4(t^2 - 2t + 8 - 3t - 4) + 10 = 4(t^2 - 5t + 4) + 10$	$\underbrace{T(t, 4-2t)}_{T_3(t)} = 4(t^2 + t(4-2t) + 2(4-2t)^2 - 3t + 2(4-2t)) + 10 = \ldots = 4(7t^2 - 35t + 40) + 10$

In each of 3 cases use Calc 1: Solve $\frac{d}{dt} T_i(t) = 0$, $i = 1,2,3$ in the corresp. open intervals

$\frac{dT_1}{dt} = 4(4t + 2) = 0 \Rightarrow t = -\frac{1}{2} \in (-2,4)$ ✓	$\frac{dT_2}{dt} = 4(2t - 5) = 0 \Rightarrow t = \frac{5}{2} \in (0,3)$ ✓	$\frac{dT_3}{dt} = 4(14t - 35) = 0$, $t = \frac{35}{14} = \frac{5}{2} \in (0,3)$

Calculate the corresp. values

$\underbrace{T_1(-\tfrac{1}{2})}_{T(0,-\frac{1}{2})} = 4(2\cdot\frac{1}{4} - 2\cdot\frac{1}{2}) + 10 = \underline{8}$

$\underbrace{T_2(\tfrac{5}{2})}_{T(\frac{5}{2},-2)} = 4(\frac{25}{4} - \frac{25}{2} + 4) + 10 = -25 + 16 + 10 = \underline{1}$

$\underbrace{T_3(\tfrac{5}{2})}_{T(\frac{5}{2}, \underbrace{4 - 2\cdot\frac{5}{2}}_{-1})} = \ldots = \underline{-5}$

Comparing all 7 values: $T_{max} = 170$ (at $(0,4)$) and $T_{min} = -6$ (at $(2,-1)$)

Local/Relative Extrema

Second derivatives test:

Suppose f'' is continuous near a and $f'(c) = 0$ (i.e. a is a critical point).	Suppose that the second partial derivatives of f are continuous near (a,b) and $\nabla f(a,b) = \mathbf{0}$ (i.e. (a,b) is a critical point). Let $\mathcal{D} = \mathcal{D}(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$
• If $f''(c) > 0$ then $f(c)$ is a local minimum.	• If $\mathcal{D} > 0$ and $f_{xx}(a,b) > 0$ (or $f_{yy}(a,b) > 0$) then $f(a,b)$ is a local minimum.
• If $f''(c) < 0$ then $f(c)$ is a local maximum.	• If $\mathcal{D} > 0$ and $f_{xx}(a,b) < 0$ (or $f_{yy}(a,b) < 0$) then $f(a,b)$ is a local maximum.
	• If $\mathcal{D} < 0$ then $f(a,b)$ is not a local extremum (saddle point).

- If $f''(c) < 0$ then $f(c)$ is a local maximum.

- If $\mathcal{D} > 0$ and $f_{xx}(a,b) < 0$ then $f(a,b)$ is a local maximum.
  or $f_{yy}(a,b) < 0$
- If $\mathcal{D} < 0$ then $f(a,b)$ is not a local extremum (saddle point).

NOTE:

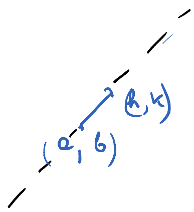
- If $f''(c) = 0$, then the test gives no information.

If $\mathcal{D} = 0$ or does not exist, then the test gives no information. fails.

To remember formula for $\mathcal{D}$:

$$\mathcal{D} = f_{xx}f_{yy} - [f_{xy}]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

$\rightarrow$ Hessian



Sketch of the proof of the Second Derivative test

$g(t) = f(a + th, b + tk)$

$g'(0) \; (= D_{(h,k)} f(a,b)) = 0$

$g''(0) \overset{?}{=} f_{xx}(a,b)\, h^2 + 2 f_{xy}(a,b)\, hk + f_{yy}(a,b)\, k^2 > 0$ for every $(h,k) \neq (0,0)$

< 0

Chain rule

$\Downarrow$ divide all by k^2 and let $s = \frac{h}{k}$

$f_{xx} s^2 + 2 f_{xy} s + f_{yy} \rightarrow$ quadratic polynom. in s

$\mathcal{D} = -\frac{1}{4}$ discriminant of this quadr. polynomial

EXAMPLE 11. *Use the Second Derivative Test to confirm that a local cold point of the lamina in the previous Example is* $(2, -1)$.

$T_x = 4(2x + y - 3) = 0$

$T_y = 4(x + 4y + 2) = 0$

prev. example $\Rightarrow$ $(x, y) = (2, -1)$ is the only crit. point

Use the second order test

$T_{xx} = 8, \quad T_{xy} = 4$

$T_{yy} = 16$

$$D = \begin{vmatrix} T_{xx} & T_{xy} \\ T_{xy} & T_{yy} \end{vmatrix}\Bigg|_{(2,-1)} = \begin{vmatrix} 8 & 4 \\ 4 & 16 \end{vmatrix} = 8 \cdot 16 - 16 = 7 \cdot 16 > 0$$

$\Downarrow$ local extremum

$T_{xx} = 8 > 0 \Rightarrow$ local minimum

EXAMPLE 12. *Find the local maximum and minimum values and saddle point(s) of the function* $f(x, y) = 4xy - x^4 - y^4$.

1. Find critical points

$$\begin{cases} f_x = 4y - 4x^3 = 0 \\ f_y = 4x - 4y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} y = x^3 \\ x = y^3 \end{cases} \Rightarrow y = (y^3)^3 = y^9$$

$x = y^3$

$$\begin{cases} f_x = 4y - 4x^3 = 0 \\ f_y = 4x - 4y^3 = 0 \end{cases} \iff \begin{cases} y = x^3 \\ x = y^3 \end{cases} \xrightarrow{x=y^3} \Rightarrow y = (y^3)^3 = y^9$$

$$y = y^9 \iff \underbrace{y^9 - y}_{\text{factor it}} = 0 \iff y(\underbrace{y^8}_{(y^4)^2} - 1) = 0 \iff y(y^4-1)(y^4+1) = y(\underbrace{y^2}_{(y^2)^2}-1)(y^2+1)(y^4+1) =$$

$a^2 - 1 = (a-1)(a+1)$

$$= y(y-1)(y+1)\underbrace{(y^2+1)}_{>0}\underbrace{(y^4+1)}_{>0} = 0 \Rightarrow \text{either } y = 0 \text{ or } y = 1 \text{ or } y = -1$$

$x = y^3 \rightarrow$ $x = 0$, $x = 1$, $x = -1$

There are 3 crit. points $(0,0)$, $(1,1)$ & $(-1,-1)$

2. Find second derivatives

$f_{xx} = -12x^2$

$f_{xy} = 4$

$f_{yy} = -12y^2$

3. Apply the second derivative test

	$(0,0)$	$(1,1)$	$(-1,-1)$
$f_{xx} = -12x^2$	0	$-12 < 0$	$-12 < 0$
$f_{xy} = 4$	4	4	4
$f_{yy} = -12y^2$	0	-12	-12
$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$	$\begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0$	$\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 > 0$	$\begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 > 0$
	saddle point	local maximum	local maximum