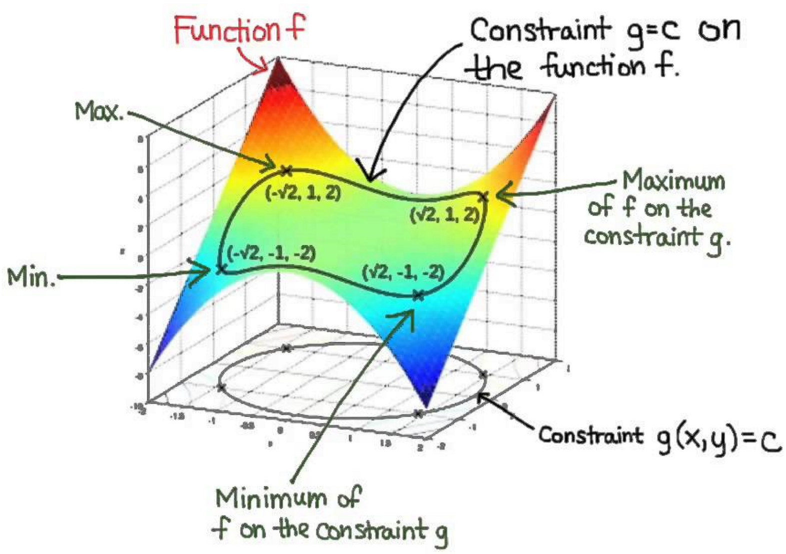
F19_LN_14_8

Friday, October 11, 2019 6:39 AM

F19_LN_1...

14.8: Lagrange Multipliers

PROBLEM: *Maximize/minimize a general function* $z = f(x, y)$ *subject to a constraint (or side condition) of the form* $g(x, y) = c$.



http://4.bp.blogspot.com/-wwTBUQGsFyQ/VqH2rKDMoNI/AAAAAAAAADM/7SD6-oKJPUM/s1600/maxresdefault.jpg

METHOD OF LAGRANGE MULTIPLIERS: *To Maximize/minimize a general function* $z = f(x, y)$ *subject to a constraint of the form* $g(x, y) = c$ *(assuming that these extreme values exist):*

1. *Find all values* x, y *and* λ *(a Lagrange multiplier) s.t.*

$$\nabla f(x, y) = \lambda \nabla g(x, y) \iff \nabla f(x,y) \parallel \nabla g(x,y)$$

and

$$g(x, y) = c$$

2. *Evaluate* f *at all points* (x, y) *that arise from the previous step. The largest of these values is the* $\max f$*; the smallest is the* $\min f$.

Rewrite the system

is the max f*; the smallest is the* min f*.*

Rewrite the system

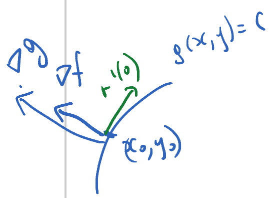
$$\nabla f(x, y) = \lambda \nabla g(x, y)$$
$$g(x, y) = c$$

in component form:.

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = c \end{cases}$$

EXPLANATION via properties of gradient

Assume that f has local extremum at (x_0, y_0) under $g(x,y) = c$



$g(x,y) = c$ is some curve in $\mathbb{R}^2$

Parametrize it: $\begin{cases} x = x(t) \\ y = y(t) \end{cases} \Leftrightarrow \vec{r} = \vec{r}(t)$ s.t. $\begin{matrix} x(0) = x_0 \\ y(0) = y_0 \end{matrix}$

$t \to f(x(t), y(t))$ has a local extremum at $t = 0$

$$\Downarrow$$

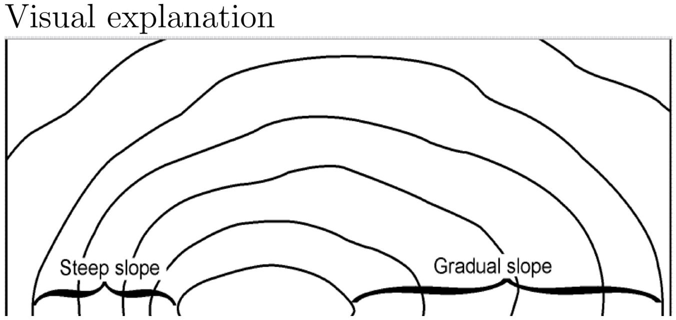
Chain rule

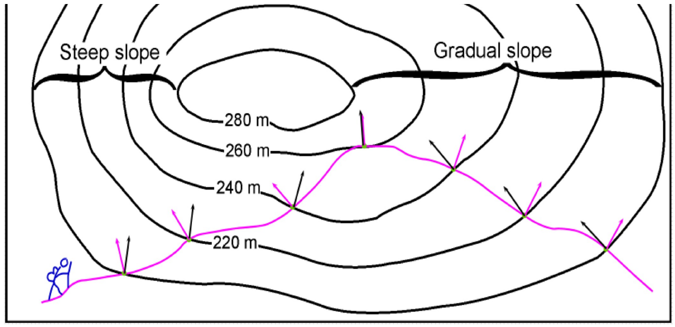
$$0 = \frac{d}{dt}\Big|_{t=0} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x_0,y_0)\, x'(0) + \frac{\partial f}{\partial y}(x_0,y_0)\, y'(0) =$$
$$= \nabla f(x_0,y_0) \cdot r'(0) = 0 \Rightarrow \nabla f \perp r'(0)$$

On the other hand, $\nabla g \perp r'(0)$, because ∇g is orthogonal to level curves

$$\begin{matrix} \nabla f|_{(x,y)} \perp r'(0) \\ \nabla g|_{x,y} \perp r'(0) \end{matrix} \Rightarrow \nabla f(x_0,y_0) \parallel \nabla g(x_0,y_0)$$

Visual explanation





https://math.stackexchange.com/questions/686538/how-to-explain-lagrange-multipliers-to- a - l a y - a u d i e n c e/686655

EXAMPLE 1. *Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x^2 + y^2$ subject to $\underbrace{x^4 + y^4}_{g(x,y)} = 1$.*

Find x, y, λ satisfying:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 1 \end{cases} \Leftrightarrow \begin{cases} 2x = \lambda \cdot 4x^3 \\ 2y = \lambda \cdot 4y^3 \\ x^4 + y^4 = 1 \end{cases} \Leftrightarrow \begin{cases} x - 2\lambda x^3 = 0 \\ y - 2\lambda y^3 = 0 \\ x^4 + y^4 = 1 \end{cases} \Leftrightarrow \begin{cases} x(1 - 2\lambda x^2) = 0 & (Eq\,1) \\ y(1 - 2\lambda y^2) = 0 & (Eq\,2) \\ x^4 + y^4 = 1 & (Eq\,3) \end{cases}$$

From the 1^{st} eq: $x = 0$ or $1 - 2\lambda x^2 = 0$

$x = 0 \overset{(eq\,3)}{\Downarrow}$ $y^4 = 1 \Rightarrow y = 1$ or $y = -1$

$y = 1$: $\Downarrow (eq\,2)$ $1 - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$ $\Downarrow$ $(x,y) = (0,1)$ is a crit. point with $\lambda = \frac{1}{2}$

$y = -1$: $\Downarrow$ Similarly $\downarrow$ $(x,y) = (0,-1)$ is a crit point with $\lambda = \frac{1}{2}$

$1 - 2\lambda x^2 = 0$ $\Downarrow$ from (Eq 2): $y = 0$ or $1 - 2\lambda y^2 = 0$

$y = 0$ $\Downarrow$ in the same as for $x = 0$ $\Downarrow$ $(x,y) = (\pm 1, 0)$ is a crit. point with $\lambda = \frac{1}{2}$

$1 - 2\lambda y^2 = 0$ $\Downarrow$
$$\begin{cases} 1 - 2\lambda x^2 = 0 & (4) \\ 1 - 2\lambda y^2 = 0 & (5) \\ x^4 + y^4 = 1 & (6) \end{cases}$$
(4) & (5) $\Rightarrow$ $x^2 = y^2 \left(= \frac{1}{2\lambda}\right)$ $\Downarrow$ plug into (6)

$2y^4 = 1 \Rightarrow y = \pm \frac{1}{\sqrt[4]{2}} = \pm x$ $\Downarrow$ 4 crit. point $\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right)$

$f = x^2 + y^2$

Calculate f in all crit. points and compare

$f(0, \pm 1) = f(\pm 1, 0) = \underline{1} \to$ min

$1 < \sqrt{2}$

$f\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f(0,\pm 1) = f(\pm 1, 0) = 1 \rightarrow$ min

crit. point $\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)$

$1 < \sqrt{2}$

$$\boxed{f\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \rightarrow \text{max}}$$

Answer: min $= 1$

max $= \sqrt{2}$