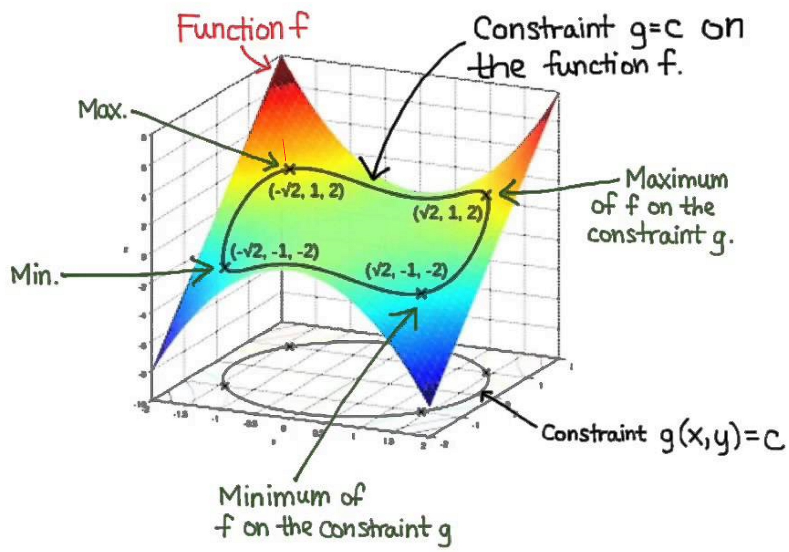
F19_LN_14_8

Friday, October 11, 2019 6:39 AM

F19_LN_1...

14.8: Lagrange Multipliers

PROBLEM: *Maximize/minimize a general function* $z = f(x, y)$ *subject to a constraint (or side condition) of the form* $g(x, y) = c$.



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METHOD OF LAGRANGE MULTIPLIERS: *To Maximize/minimize a general function* $z = f(x, y)$ *subject to a constraint of the form* $g(x, y) = c$ *(assuming that these extreme values exist):*

1. *Find all values* x, y *and* λ *(a Lagrange multiplier) s.t.*

$$\nabla f(x, y) = \lambda \nabla g(x, y) \iff \nabla f(x,y) \parallel \nabla g(x,y)$$

and

$$g(x, y) = c$$

2. *Evaluate* f *at all points* (x, y) *that arise from the previous step. The largest of these values is the* max f*; the smallest is the* min f.

Rewrite the system

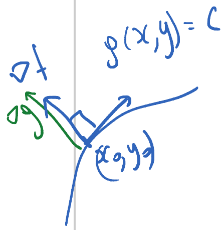
is the max f*; the smallest is the* min f.

Rewrite the system

$$\nabla f(x,y)=\lambda\nabla g(x,y)$$
$$g(x,y)=c$$

in component form:.

$$\begin{cases} f_x(x,y)=\lambda\, g_x(x,y)\\ f_y(x,y)=\lambda\, g_y(x,y)\\ g(x,y)=c\end{cases} \to \text{3 eq. \& 3 unknowns } (x,y,\lambda)$$



EXPLANATION via properties of gradient

Assume that f has a local max or min at (x_0,y_0) under contrain $g(x,y)=c$

Parametrize $g(x,y)=c$ by

$$\begin{cases} x=x(t)\\ y=y(t)\end{cases} \iff \vec{r}=\vec{r}(t) \quad \text{s.t.} \quad x(0)=x_0,\ y(0)=y_0$$

$t\to f(x(t),y(t))$ has local extremum at $t=0$

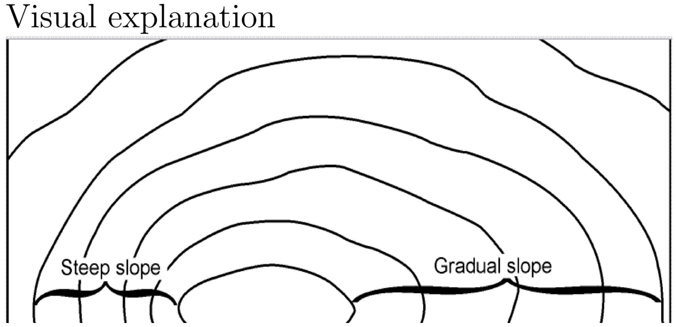
$\Downarrow$

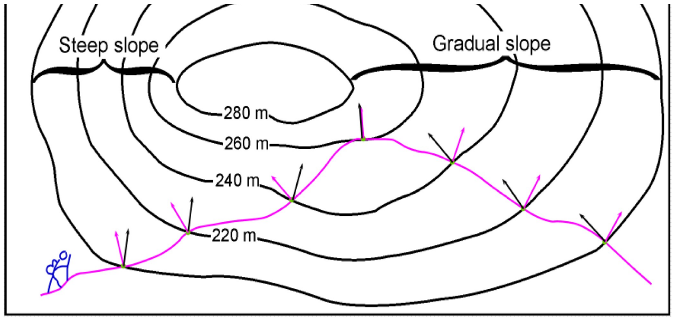
$$0=\frac{d}{dt}f(x(t),y(t))\Big|_{t=0} \overset{\text{chain rule}}{=} f_x(x_0,y_0)\,x'(0)+f_y(x_0,y_0)\,y'(0)=$$
$$=\nabla f(x_0,y_0)\cdot r'(0)=0 \iff \nabla f(x_0,y_0)\perp r'(0)$$

On the other hand ∇g is orthogonal to $g(x,y)=c$ (it is the level curve of g), i.e. $\nabla g(x_0,y_0)\perp r'(0)$

$$\left.\begin{array}{l}\nabla f(x_0,y_0)\perp r'(0)\\ \nabla g(x_0,y_0)\perp r'(0)\end{array}\right\} \Rightarrow \nabla f(x_0,y_0)\parallel \nabla g(x_0,y_0) \Rightarrow \text{there exists } \lambda \text{ s.t. } \nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$$

Visual explanation





https://math.stackexchange.com/questions/686538/how-to-explain-lagrange-multipliers-to- a - l a y - a u d i e n c e/686655

EXAMPLE 1. *Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x^2 + y^2$ subject to $\underbrace{x^4 + y^4}_{g(x,y)} = 1$.* $g(x,y) = x^4 + y^4$

1) Find x, y, and λ s.t

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 1 \end{cases} \iff \begin{cases} 2x = 4\lambda x^3 \\ 2y = 4\lambda y^3 \\ x^4 + y^4 = 1 \end{cases} \iff \begin{cases} x - 2\lambda x^3 = 0 \\ y - 2\lambda y^3 = 0 \\ x^4 + y^4 = 1 \end{cases} \iff \begin{cases} x(1-2\lambda x^2) = 0 & (1) \\ y(1-2\lambda y^2) = 0 & (2) \\ x^4 + y^4 = 1 & (3) \end{cases}$$

Eq (1): $x(1-2\lambda x^2) = 0 \Rightarrow$ Either $x = 0$ or $1 - 2\lambda x^2 = 0$

Either $x=0$:

$\Downarrow$ Plug in (3)

$y^4 = 1 \Rightarrow y = \pm 1$

$\Downarrow$ Plug in (2)

$1 - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$

$\Downarrow$

$(0, \pm 1)$ are a crit. point (with $\lambda = \frac{1}{2}$)

or $1-2\lambda x^2=0$:

From Eq (2):

$y(1-2\lambda y^2) = 0 \Rightarrow$

Either $y = 0$ or $1 - 2\lambda y^2 = 0$

$\Downarrow$ By analogy with the case $x=0$

we get that $(\pm 1, 0)$ are crit points (with $\lambda = \frac{1}{2}$)

$\Downarrow$

$$\begin{cases} 1-2\lambda x^2 = 0 & (4) \\ 1-2\lambda y^2 = 0 & (5) \\ x^4 + y^4 = 1 & (6) \end{cases}$$

$(4) \& (5) \Rightarrow$

$x^2 = y^2 \ (= \frac{1}{2\lambda})$

$\Downarrow$ Plug to (6) $x^4 = y^4$

$2y^4 = 1 \iff$

$y^4 = \frac{1}{2} \Rightarrow$

$f(x,y) = x^2 + y^2$

2) Evaluate f at all 8 crit. points and compare

$f(\pm 1, 0) = f(0, \pm 1) = \underline{1} \longrightarrow$ min

$f\left(\pm \frac{1}{\sqrt[4]{2}}, \pm \frac{1}{\sqrt[4]{2}}\right) = \left(\pm\frac{1}{\sqrt[4]{2}}\right)^2 + \left(\pm\frac{1}{\sqrt[4]{2}}\right)^2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} > 1$

$$f\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = \left(\pm\frac{1}{\sqrt[4]{2}}\right)^2 + \left(\pm\frac{1}{\sqrt[4]{2}}\right)^2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \underset{\substack{\downarrow \\ \text{max}}}{\sqrt{2}} > 1$$

$2y^4 = 1 \iff$

$y^4 = \frac{1}{2} \Rightarrow$

$y = \pm\frac{1}{\sqrt[4]{2}} \Rightarrow$

$x = \pm y = \pm\frac{1}{\sqrt[4]{2}}$

$\Downarrow$

$\underbrace{\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)}_{4\text{ point}}$ are critical points

Answer: $\min_{\{g=1\}} f = \boxed{1}$

$\max_{\{g=1\}} f = \boxed{\sqrt{2}}$