F19_LN_15_1

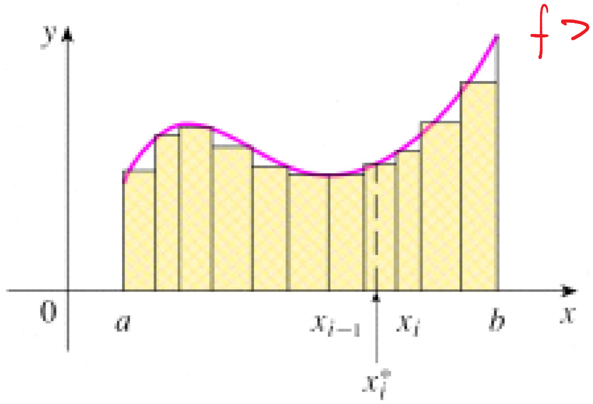
Tuesday, October 15, 2019 8:19 PM

F19_LN_1...

15.1: Double integrals over rectangles

Recall that a single definite integral can be interpreted as **area**:

$f > 0$



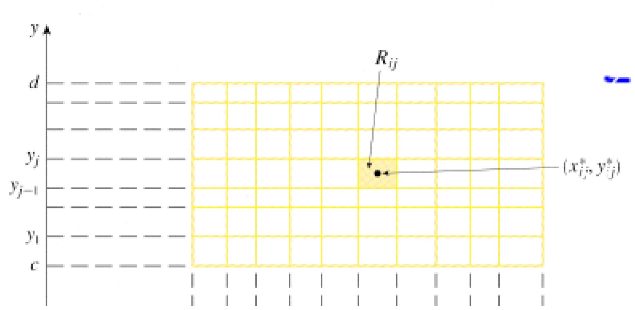
The exact area is also the definition of the definite integral:

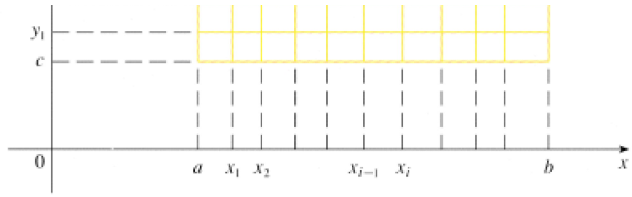
$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

Problem: Assume that $f(x, y)$ is defined on a closed rectangle

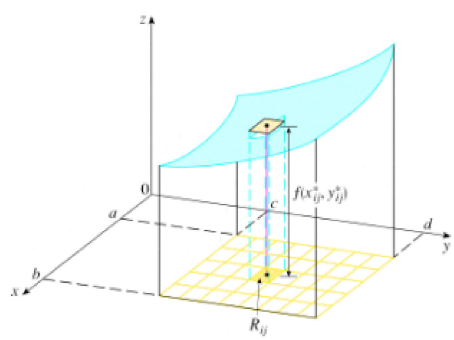
$R = [a, b] \times [b, c] = \{(x, y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$ and $f(x, y) \ge 0$ over R. Denote by S the part of the surface $z = f(x, y)$ over the rectangle R. What the volume of the region under S and above the xy-plane is?

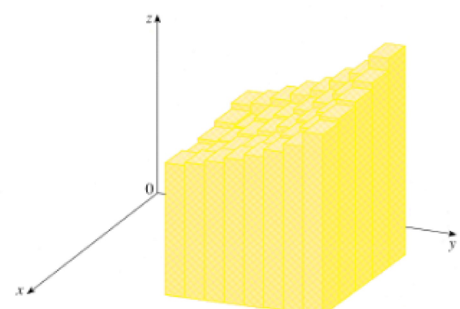
Solution: Approximate the volume. Divide up $a \le x \le b$ into n subintervals and divide up $c \le y \le d$ into m subintervals. From each of these smaller rectangles choose a point (x_i^*, y_j^*).





Over each of these smaller rectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$.





The volume is given by

$$\lim_{n,m\to\infty} \overbrace{\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x\Delta y}^{\text{Riemann sum}}$$

which is also the definition of a double integral

$$\iint_R f(x,y)\mathrm{d}A.$$

Another notation: $\displaystyle\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y.$

THEOREM 1. *If f is continuous on R then f is integrable over R.*

THEOREM 2. *If $f(x,y) \geq 0$ and f is continuous on the rectangle $R = [a,b]\times[c,d]$, then the volume V of the solid S that lies above R and inder the graph of f, i.e.*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 | (x,y) \in R, 0 \leq z \leq f(x,y), (x,y) \in R\right\},$$

is

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

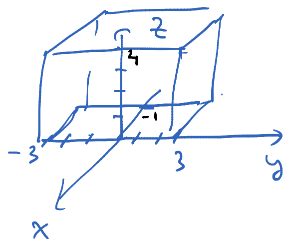
is

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

EXAMPLE 3. *Evaluate the integral*

$$\iint_R 4\,\mathrm{d}A$$

where $R = [-1,0]\times[-3,3]$ *by identifying it as a volume of a solid.*



The solid above the rectangle R and below the graph $z=4$ (here $f(x,y)=4$)

The solid is the box of size $1\times 6\times 4$, i.e. the volume $=\boxed{24}$

Iterated integrals

Suppose that $f(x,y)$ is integrable over the rectangle $R=[a,b]\times[b,c]$.

Partial integration of f with respect to x: $\int_a^b f(x,y)\,\mathrm{d}x$

Partial integration of f with respect to y: $\int_c^d f(x,y)\,\mathrm{d}y$

EXAMPLE 4.

$$\int_0^4 (x+3y^2)\,\mathrm{d}x = \frac{x^2}{2} + 3y^2x\Big|_{x=0}^{4} = \frac{16}{2} + 3y^2\cdot 4 - 0 = 8 + 12y^2$$

$$\int_1^4 e^{xy}\,\mathrm{d}y = \underset{x \text{ is constant}}{\frac{1}{x}e^{xy}}\Big|_{y=1}^{4} = \frac{1}{x}e^{4x} - \frac{1}{x}e^{x} = \frac{1}{x}\left(e^{4x}-e^{x}\right)$$

Iterated integrals:

$$\int_a^b \Bigg[\overbrace{\int_c^d f(x,y)\,\mathrm{d}y}^{\text{function of } x}\Bigg]\mathrm{d}x = \int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x$$

and

$$\int_c^d \Bigg[\overbrace{\int_a^b f(x,y)\,\mathrm{d}x}^{\text{function of } y}\Bigg]\mathrm{d}y = \int_c^d\int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 5. *Evaluate the integrals:*

$$I_1 = \int_0^{\ln 2}\int_0^{\ln 5} \underbrace{e^{2x-y}}_{e^{2x}e^{-y}}\,\mathrm{d}y\mathrm{d}x, \quad I_2 = \int_0^{\ln 5}\int_0^{\ln 2} e^{2x-y}\,\mathrm{d}x\mathrm{d}y \quad = \int_0^{\ln 5} e^{-y}\mathrm{d}y \cdot \int_0^{\ln 2} e^{2x}\mathrm{d}x$$

constant

$$I_1 = \int^{\ln 2}\int^{\ln 5} e^{2x}e^{-y}\,\mathrm{d}y\,\mathrm{d}x = \int^{\ln 2} e^{2x}\Big(\int^{\ln 5} e^{-y}\,\mathrm{d}y\Big)\mathrm{d}x =$$

$e^{2x}e^{-y}$ constant

$$I_1 = \int_0^{\ln 2}\int_0^{\ln 5} \underbrace{e^{2x}}_{\text{a constant}} e^{-y}\,dy\,dx = \int_0^{\ln 2} e^{2x}\left(\int_0^{\ln 5} e^{-y}\,dy\right)dx =$$

$$= \int_0^{\ln 2} e^{2x}\left(-e^{-y}\Big|_{y=0}^{\ln 5}\right)dx = \int_0^{\ln 2} e^{2x}\left(\overbrace{\underbrace{-e^{-\ln 5}}_{-e^{\ln\frac{1}{5}} = -\frac{1}{5}} + \underbrace{e^0}_{1}}^{1-\frac{1}{5}}\right)dx =$$

$$= \left(1-\frac{1}{5}\right)\int_0^{\ln 2} e^{2x}\,dx = \frac{\cancel{4}^{\,2}}{5}\cdot\frac{1}{\cancel{2}}e^{2x}\Big|_{x=0}^{\ln 2} = \frac{2}{5}\cdot\left(e^{2\ln 2} - e^0\right) = \frac{2}{5}(4-1) = \frac{6}{5}$$

Similarly $I_2 = \int_0^{\ln 5} e^{-y}\,dy \cdot \int_0^{\ln 2} e^{2x}\,dx = I_1 = \frac{6}{5}$

FUBINI's THEOREM: *If f is continuous ~~on~~ on the rectangle $R = [a,b]\times[c,d]$ then*

$$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x = \int_c^d\int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

The order of integration is not important

EXAMPLE 6. *Make a conclusion from the Example 5 based on the Fubini Theorem.*

$$\iint_R e^{2x-y}\,dA = \frac{6}{5}$$

$R = [0, \ln 2]\times[0, \ln 5]$

COROLLARY 7. *If g and h are continuous functions of one variable and $R = [a,b]\times[c,d]$ then*

$$\iint_R g(x)h(y)\,\mathrm{d}A = \left(\int_a^b g(x)\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

$$\iint_R g(x)h(y)\,dA \overset{\text{Fubini}}{=} \int_a^b\int_c^d \underbrace{g(x)}_{\substack{\text{constant}\\ \text{when integrating}\\ \text{w.r.t. } y}} h(y)\,dy\,dx = \int_a^b g(x)\Big(\underbrace{\int_c^d h(y)\,dy}_{\text{Constant}}\Big)dx =$$

$$= \int_c^d h(y)\,dy \int_a^b g(x)\,dx \quad ▨$$

In our previous example $e^{2x-y} = \underbrace{e^{2x}}_{g(x)}\cdot\underbrace{e^{-y}}_{h(y)}$

EXAMPLE 8. *Evaluate*

$$\iint_R x\cos(xy)\,\mathrm{d}A \overset{\text{By Fubini}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_1^5 x\cos(xy)\,dy\right)dx \to \text{more simple}$$

$$= \int_1^5\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos(xy)\,dx\right)dy$$

EXAMPLE 8. *Evaluate*

$$\iint_R x\cos(xy)\,\mathrm{d}A$$

where $R = [-\pi/2, \pi/2] \times [1, 5]$ *and describe your result geometrically.*

By

$$= \int_{-\frac{\pi}{2}} \left(\int_1 x\cos(xy)\,dy \right) dx \to \text{simple}$$

$$= \int_1^5 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos(xy)\,dx \right) dy$$

here you need integration by parts

$$\iint_R x\cos(xy)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_1^5 x\cos(xy)\,dy \right) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \left(\int_1^5 \cos(xy)\,dy \right) dx =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cancel{x} \left(\frac{1}{\cancel{x}} \sin(xy) \Big|_{y=1}^{5} \right) dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin 5x - \sin x)\,dx =$$

$$= -\frac{1}{5}\cos 5x \Big|_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} - (-\cos x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$$

cos x is an even function: $\cos(-x) = \cos x$

EXAMPLE 9. *Express the volume of the solid* S *lying under the circular paraboloid* $z = x^2 + y^2$ *and above the rectangle* $R = [-2, 2] \times [-3, 3]$ *using an iterated integral.*

$$V = \iint_R (x^2+y^2)\,dx\,dy = \int_{-2}^{2} \left(\int_{-3}^{3} (x^2+y^2)\,dy \right) dx =$$

$$= \int_{-2}^{2} \left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=-3}^{3} dx = 2\int_{-2}^{2} \left(3x^2 + \frac{27}{3} \right) dx = 2\left(x^3 + 9x \right)\Big|_{x=-2}^{2} =$$

$$= 2 \cdot 2 (2^3 + 9 \cdot 2) = 4(8+18) = \boxed{104}$$