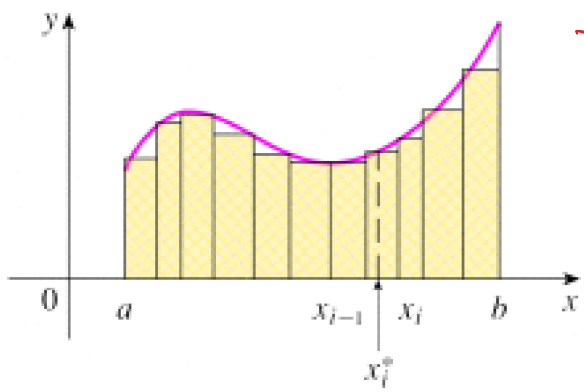
F19_LN_15_1

Tuesday, October 15, 2019 8:20 PM

F19_LN_1...

15.1: Double integrals over rectangles

Recall that a single definite integral can be interpreted as **area**:



$f(x) > 0$

The exact area is also the definition of the definite integral:

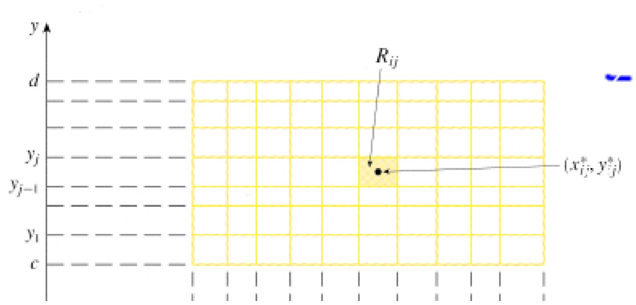
$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

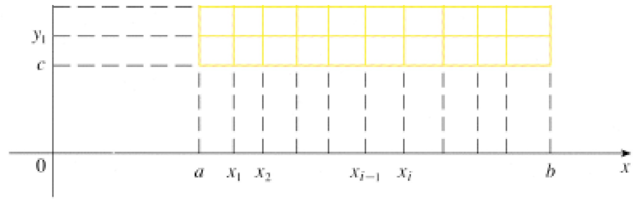
Riemann sum

Problem: Assume that $f(x,y)$ is defined on a closed rectangle

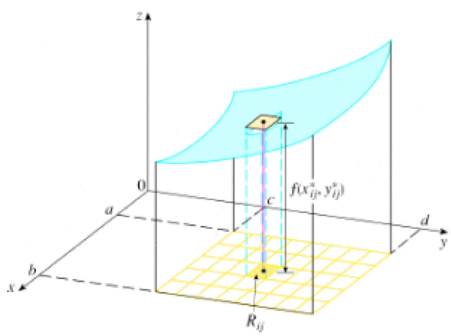
$R = [a,b] \times [b,c] = \{(x,y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$ and $f(x,y) \ge 0$ over R. Denote by S the part of the surface $z = f(x,y)$ over the rectangle R. What the volume of the region under S and above the xy-plane is?

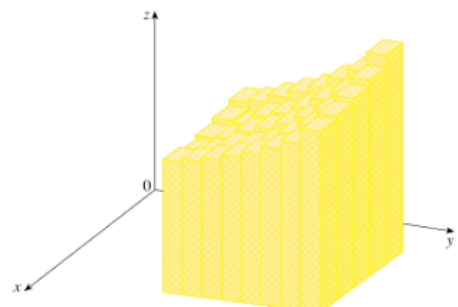
Solution: Approximate the volume. Divide up $a \le x \le b$ into n subintervals and divide up $c \le y \le d$ into m subintervals. From each of these smaller rectangles choose a point (x_i^*, y_j^*).





Over each of these smaller rectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$.





The volume is given by

$$\lim_{n,m\to\infty} \overbrace{\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x \Delta y}^{\text{Riemann sum}}$$

which is also the definition of a double integral

$$\iint_R f(x,y)\mathrm{d}A.$$

Another notation: $\displaystyle\iint_R f(x,y)\,\mathrm{d}A = \iint_R f(x,y)\,\mathrm{d}x\mathrm{d}y.$

THEOREM 1. *If f is continuous on R then f is integrable over R.*

THEOREM 2. *If $f(x,y) \geq 0$ and f is continuous on the rectangle $R = [a,b]\times[c,d]$, then the volume V of the solid S that lies above R and inder the graph of f, i.e.*

$$S = \left\{(x,y,z) \in \mathbb{R}^3 | (x,y) \in R, 0 \leq z \leq f(x,y), (x,y) \in R\right\},$$

is

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

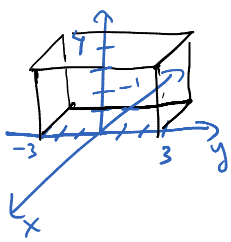
is

$$V = \iint_R f(x,y)\,\mathrm{d}A.$$

EXAMPLE 3. *Evaluate the integral*

$$\iint_R 4\,\mathrm{d}A$$

where $R = [-1,0]\times[-3,3]$ by identifying it as a volume of a solid.



$f(x,y) \equiv 4$

The solid S is above R and below the graph of $z = 4$ $\Rightarrow$

S = the box with sizes $1\times 6\times 4$ $\Rightarrow$

$V = 6\cdot 4 = \boxed{24}$

Iterated integrals

Suppose that $f(x,y)$ is integrable over the rectangle $R = [a,b]\times[b,c]$.

Partial integration of f with respect to x: $\int_a^b f(x,\underset{\text{fixed}}{y})\,\mathrm{d}x$ $\longrightarrow$ a function y

Partial integration of f with respect to y: $\int_c^d f(\underset{\text{fixed}}{x},y)\,\mathrm{d}y$ $\longrightarrow$ a function of x

EXAMPLE 4.

$$\int_0^4 (x+3\underset{\text{fixed}}{y^2})\,\mathrm{d}x = \frac{x^2}{2}+3y^2x\Big|_{x=0}^{4} = \frac{4^2}{2}+3\cdot y^2\cdot 4 - 0 = 8+12y^2$$

$$\int_1^4 e^{\underset{\text{fixed}}{x}y}\,\mathrm{d}y = \frac{1}{x}e^{xy}\Big|_{y=1}^{4} = \frac{1}{x}e^{4x}-\frac{1}{x}e^{x} = \frac{1}{x}\left(e^{4x}-e^{x}\right)$$

Iterated integrals:

$$\underbrace{\int_a^b \overbrace{\left[\int_c^d f(x,y)\,\mathrm{d}y\right]}^{\text{a function of } x}\mathrm{d}x}_{\text{a number}} = \int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x$$

and

$$\int_c^d \left[\int_a^b f(x,y)\,\mathrm{d}x\right]\mathrm{d}y = \int_c^d\int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 5. *Evaluate the integrals:*

$$I_1 = \int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,\mathrm{d}y\mathrm{d}x,\quad I_2 = \int_0^{\ln 5}\int_0^{\ln 2} e^{2x-y}\,\mathrm{d}x\mathrm{d}y$$

$$I = \int^{\ln 2}\left(\int^{\ln 5} e^{2x}e^{-y}\,dy\right)dx = \int^{\ln 2} e^{2x}\left(\int^{\ln 5} e^{-y}\,dy\right)dx = \int_0^{\ln 5} e^{-y}dy\int_0^{\ln 2} e^{2x}dx$$

a constant

$$I_1 = \int_0^{\ln 2}\Big(\int_0^{\ln 5} \underbrace{e^{2x}}_{\text{a constant as we integrate w.r.t. } y} e^{-y}\,dy\Big)dx = \int_0^{\ln 2} e^{2x}\Big(\underbrace{\int_0^{\ln 5} e^{-y}\,dy}_{\text{a constant}}\Big)dx = \int_0^{\ln 5} e^{-y}dy \int_0^{\ln 2}$$

$$= \int_0^{\ln 2} e^{2x}\Big(-e^{-y}\Big|_{y=0}^{\ln 5}\Big)dx = \int_0^{\ln 2} e^{2x}\Big(\overbrace{-\underbrace{e^{-\ln 5}}_{(e^{\ln 5})^{-1}=\frac{1}{5}} - (\underbrace{-e^{0}}_{-1})}^{1-\frac{1}{5}=\frac{4}{5}}\Big)dx = \frac{4}{5}\int_0^{\ln 2} e^{2x}dx =$$

$$= \frac{\cancel{4}^{\,2}}{5}\,\frac{1}{\cancel{2}}e^{2x}\Big|_{x=0}^{\ln 2} = \frac{2}{5}\Big(\underbrace{e^{2\ln 2}}_{(e^{\ln 2})^2=2^2=4} - \overbrace{e^{0}}^{1}\Big) = \frac{2}{5}\cdot 3 = \boxed{\frac{6}{5}}$$

In the same way $I_2 = \int_0^{\ln 2} e^{2x}dx \int_0^{\ln 5} e^{-y}dy = I_1 = \frac{6}{5}$

FUBINI's THEOREM: *If f is continuous on on the rectangle $R = [a, b] \times [c, d]$ then*

$$\iint_R f(x, y)\,\mathrm{d}A = \int_a^b\int_c^d f(x, y)\,\mathrm{d}y\mathrm{d}x = \int_c^d\int_a^b f(x, y)\,\mathrm{d}x\mathrm{d}y.$$

the order of integration in the iterated integrals is not important

EXAMPLE 6. *Make a conclusion from the Example 5 based on the Fubini Theorem.*

$$\iint_R e^{2x-y}dA = \frac{6}{5} \quad \text{where} \quad R = [0, \ln 2] \times [0, \ln 5]$$

COROLLARY 7. *If g and h are continuous functions of one variable and $R = [a, b] \times [c, d]$ then*

$$\iint_R g(x)h(y)\,\mathrm{d}A = \left(\int_a^b g(x)\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right).$$

$$\iint_R g(x)\,h(y)\,dA \overset{\text{Fubini}}{=} \int_a^b\int_c^d \underbrace{g(x)}_{\text{a constant when integrating w.r.t. } y} h(y)\,dy\,dx = \int_a^b g(x)\Big(\underbrace{\int_c^d h(y)\,dy}_{\text{a constant}}\Big)dx =$$

$$= \int_c^d h(y)\,dy \int_a^b g(x)\,dx$$

In the previous example $e^{2x-y} = \underbrace{e^{2x}}_{g(x)}\underbrace{e^{-y}}_{h(y)}$

EXAMPLE 8. *Evaluate*

$$\iint_R x\cos(xy)\,\mathrm{d}A \overset{\text{Fubini}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\int_1^5 x\cos(xy)\,dy\Big)dx \to \text{more easy}$$

$$= \int_1^5 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

EXAMPLE 8. *Evaluate*

$$\iint_R x\cos(xy)\,\mathrm{d}A$$

where $R = [-\pi/2, \pi/2] \times [1,5]$ *and describe your result geometrically.*

Fubini $= \int_{-\frac{\pi}{2}}\left(\int_1 x\cos(xy)\,dy\right)dx \rightarrow$ more easy

$= \int_1^5\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos(xy)\,dx\right)dy$

one needs to use integration by parts

$$\iint_R x\cos(xy)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_1^5 x\cos xy\,dy\right)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\left(\int_1^5 \cos xy\,dy\right)dx =$$

constant when integrating w.r.t. y

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cancel{x}\,\frac{1}{\cancel{x}}\sin xy\Big|_{y=1}^{5}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin 5x - \sin x)\,dx = -\frac{1}{5}\cos 5x\Big|_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} + \cos x\Big|_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= 0$$

cos is even, i.e. $\cos(-x) = \cos x$

EXAMPLE 9. *Express the volume of the solid* S *lying under the circular paraboloid* $z = \underbrace{x^2 + y^2}_{f(x,y)}$ *and above the rectangle* $R = [-2,2] \times [-3,3]$ *using an iterated integral.*

$$V = \iint_R (x^2+y^2)\,dA = \int_{-2}^{2}\int_{-3}^{3}(x^2+y^2)\,dy\,dx = \int_{-2}^{2}\left(x^2y + \frac{y^3}{3}\right)\Big|_{y=-3}^{3}dx =$$

$$= 2\int_{-2}^{2}\left(3x^2 + \overbrace{\frac{27}{3}}^{9}\right)dx = 2\,(x^3 + 9x)\Big|_{x=-2}^{2} = 2\cdot 2\,(2^3 + 9\cdot 2) = 4\cdot 26 = \boxed{104}$$