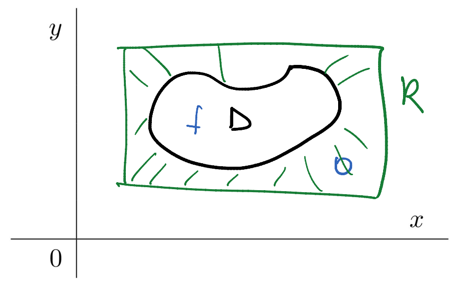
F19_LN_15_2

Sunday, October 20, 2019 9:24 PM

F19_LN_1...

15.2: Double Integrals over General Regions

`All functions below are continuous on their domains.`



Let D be a bounded region enclosed in a rectangular region R such that its boundary ∂D is sufficiently nice, for example, is a piecewise differentiable curve. We define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$

If F is integrable over R, then we say F is *integrable* over D and we define **the double integral of f over D** by

$$\iint_D f(x,y)\,\mathrm{d}A \overset{\text{def}}{=} \iint_R F(x,y)\,\mathrm{d}A$$

FACT: If $f(x,y) \geq 0$ and f is continuous on the region D then the volume V of the solid S that lies above D and under the graph of f, i.e.

$$S = \left\{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x,y), (x,y) \in D\right\},$$

is

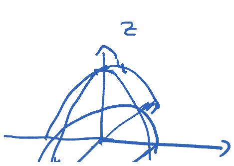
$$V = \iint_D f(x,y)\,\mathrm{d}A.$$

EXAMPLE 1. *Evaluate the integral*

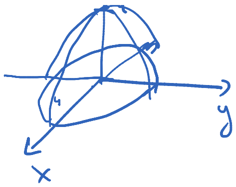
$$\iint_D \overbrace{\sqrt{16 - x^2 - y^2}}^{f}\,\mathrm{d}A$$

where $D = \left\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 16\right\}$ by identifying it as a volume of a solid.

$z = f(x,y) = \sqrt{16-x^2-y^2} \Rightarrow \underbrace{x^2+y^2+z^2=16 \ \& \ z \geq 0}_{\text{hemisphere}}$



$\iint \sqrt{}$ | the volume $1 \ 4 \ \pi \ 4^3 \ 128\pi$



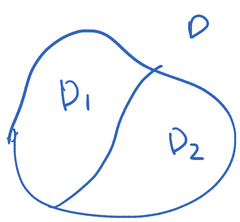
hemisphere

$$\iint \sqrt{16-x^2-y^2} = \frac{1}{2} \text{ the volume of the ball of radius 4} = \frac{1}{2}\,\frac{4}{3}\pi \cdot 4^3 = \frac{128\pi}{3}$$

Properties of double integrals:

- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps their boundaries then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$



- If α and β are real numbers then

$$\iint_D (\alpha f(x,y) + \beta g(x,y))\,\mathrm{d}A = \alpha \iint_D f(x,y)\,\mathrm{d}A + \beta \iint_D g(x,y)\,\mathrm{d}A.$$

- If we integrate the constant function $f(x,y) = 1$ over D, we get **area** of D:

$$\iint_D 1\,\mathrm{d}A = A(D).$$

a disk of radius 5

EXAMPLE 2. *If* $D = \{(x,y) \mid x^2 + y^2 \le 25\}$ *then*

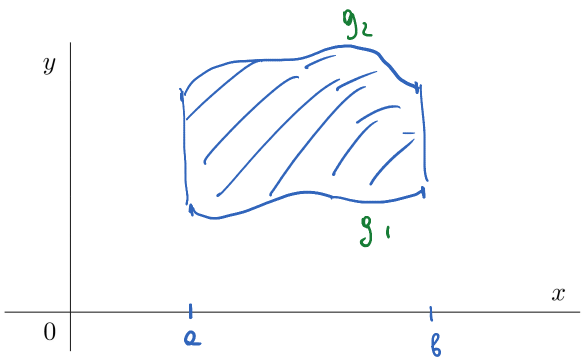
EXAMPLE 2. *If* $D = \{(x, y) |\ x^2 + y^2 \leq 25\}$ *then*

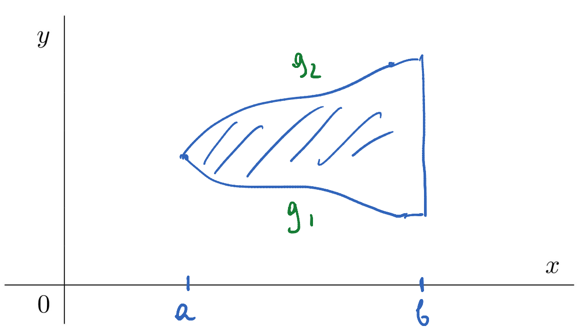
$$\iint_D \mathrm{d}A = \text{the area of } D = \pi \cdot 5^2 = 25\pi$$

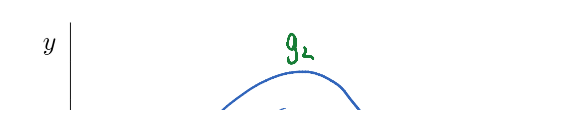
Computation of double integral:

A plain region of **TYPE I**:

$$D = \{(x, y) |\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

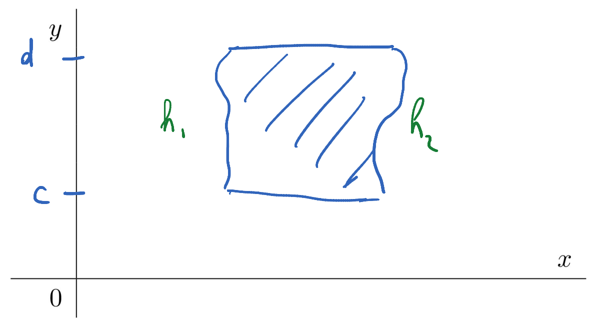


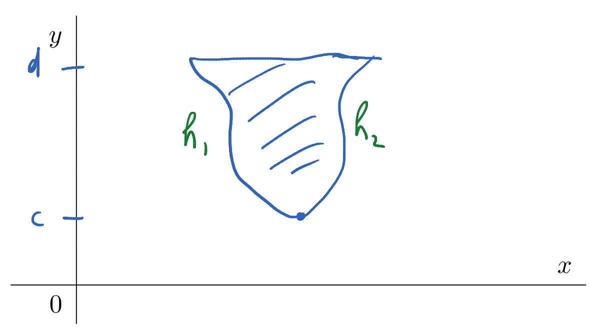


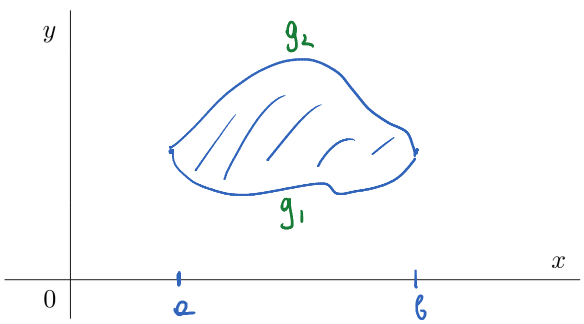


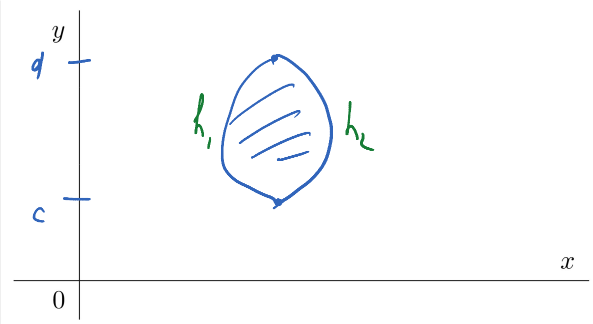
A plain region of **TYPE II**:

$$D = \{(x, y) |\ c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}.$$









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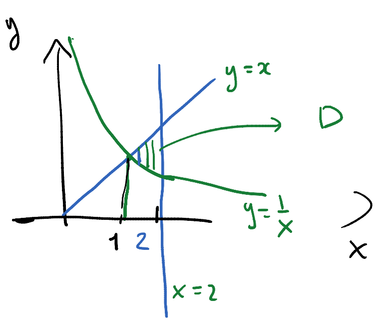
THEOREM 3. *If D is a region of type I such that $D = \{(x,y)|\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{\underbrace{g_1(x)}_{\text{lower curve}}}^{\overbrace{g_2(x)}^{\text{upper curve}}} f(x,y)\,\mathrm{d}y\mathrm{d}x.$$

THEOREM 4. *If D is a region of type II s.t. $D = \{(x,y)|\ c \le y \le d, h_1(y) \le x \le h_2(y)\}$ then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_c^d \int_{\underbrace{h_1(y)}_{\text{left curve}}}^{\overbrace{h_2(y)}^{\text{right curve}}} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

EXAMPLE 5. *Evaluate $I = \iint_D 30x^2y\,\mathrm{d}A$, where D is the region bounded by the lines $x = 2, y = x$ and the hyperbola $\underbrace{xy = 1}_{\text{hyperbola}}$ in two different ways (i.e. considering D as a type I and then as a type II region).*



1) D as a region of type 1

The lower curve is $y = \underbrace{\frac{1}{x}}_{g_1(x)}$

The upper curve is $y = \underbrace{x}_{g_2(x)}$

$b = 2$; a?

$x = \frac{1}{x}$, $x > 0 \Rightarrow x = 1$

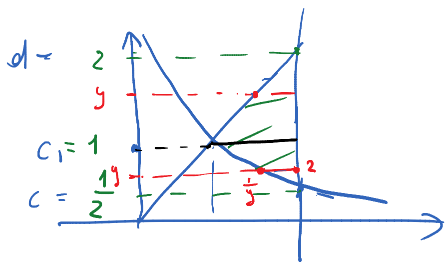
$a = 1$

$$I = \int_1^2 \left(\int_{\frac{1}{x}}^{x} 30x^2y\ dy \right) dx = 30 \int_1^2 x^2 \left(\int_{\frac{1}{x}}^{x} y\, dy \right) dx = 30 \int_1^2 x^2 \left(\frac{y^2}{2} \Big|_{y=\frac{1}{x}}^{x} \right) dx =$$

$$= \underbrace{\frac{30}{2}}_{15}\int_1^2 x^2\left(x^2-\frac{1}{x^2}\right)dx = 15\int_1^2 (x^4-1)\,dx = 15\left(\frac{x^5}{5}-x\right)\Big|_{x=1}^{2} =$$

$$= 15\left(\frac{32}{5}-2-\left(\frac{1}{5}-1\right)\right) = 15\left(\frac{31}{5}-1\right) = \overset{3}{\cancel{15}}\cdot\frac{26}{\cancel{5}} = \boxed{78}$$

D as a region of type II



c? $\begin{cases} y=\frac{1}{x} \\ x=2 \end{cases} \Rightarrow y=\frac{1}{2}$ $\quad c=\frac{1}{2}$

d? $\begin{cases} y=x \\ x=2 \end{cases} \Rightarrow y=2$ $\quad d=2$

The left curve consist of 2 curves — we have to brake D into 2 regions: c_1? $\begin{cases} y=\frac{1}{x} \\ y=x \\ x>0 \end{cases} \Rightarrow \frac{1}{x}=x \Rightarrow x=1 \Rightarrow y=1$

$xy=1 \Rightarrow x=\frac{1}{y}$

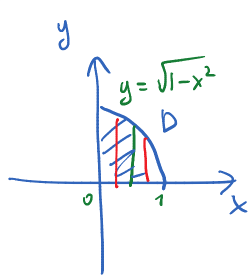
The left curve is the graph of $h_1(y)=\begin{cases} \frac{1}{y} & \frac{1}{2}\le y\le 1 \\ y & 1\le y\le 2 \end{cases}$

The right curve is $h_2(y)=2$

$\Rightarrow D=\left\{(x,y): \begin{matrix}\frac{1}{2}\le y\le 1 \\ \frac{1}{y}\le x\le 2\end{matrix}\right\} \cup \left\{(x,y): \begin{matrix}1\le y\le 2 \\ y\le x\le 2\end{matrix}\right\}$

$$\Rightarrow I = \int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{y}}^{2} 30x^2y\,dx\right)dy + \int_1^2\left(\int_y^2 30x^2y\,dx\right)dy = \dots = 78$$

EXAMPLE 6. *Find the volume of the solid bounded by the cylinder $x^2+y^2=1$ and the planes $\cancel{x=0}$, $y=z$, $z=0$ in the first octant.*

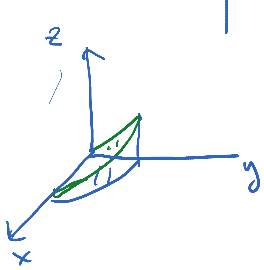


$z=f(x,y)=y$

$$V=\iint_D y\,dx\,dy = \int_0^1\left(\int_0^{\sqrt{1-x^2}} y\,dy\right)dx =$$

quater of a circle $\begin{cases} x^2+y^2\le 1 \\ x\ge 0 \\ y\ge 0 \end{cases}$

$D \xrightarrow{\text{as a region of type 1}} = \{(x,y): 0\le x\le 1,\ 0\le y\le \sqrt{1-x^2}\}$

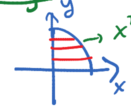


$$= \int_0^1 \frac{y^2}{2}\Big|_{y=0}^{\sqrt{1-x^2}}dx = \frac{1}{2}\int_0^1\left(\underbrace{\left(\sqrt{1-x^2}\right)^2}_{1-x^2}-0\right)dx =$$

$$= \frac{1}{2}\int_0^1 (1-x^2)\,dx = \frac{1}{2}\left(x-\frac{x^3}{3}\right)\Big|^1 = \frac{1}{2}\left(1-\frac{1}{3}\right)=\frac{1}{3}$$

$$= \frac{1}{2}\int_0^1 (\underbrace{1-x^2}) \, dx = \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Big|_{x=0}^{1} = \frac{1}{2}\underbrace{\left(1-\frac{1}{3}\right)}_{\frac{2}{3}} = \frac{1}{3}$$

Way 2: D as type 2



$x^2+y^2=1 \iff x=\sqrt{1-y^2}$, $x,y \geq 0$

$D = \{(x,y) : 0 \leq y \leq 1, \ 0 \leq x \leq \sqrt{1-y^2}\}$

$$V = \int_0^1 \int_0^{\sqrt{1-y^2}} y \, dx \, dy = \int_0^1 y\left(\int_0^{\sqrt{1-y^2}} dx\right) dy = \int_0^1 y\sqrt{1-y^2}\, dy = -\frac{1}{2}\int_1^0 \sqrt{u}\, du$$

$u = 1-y^2$: $y: 0 \to 1$, $u: 1 \to 0$

$du = -2y\,dy \Rightarrow y\,dy = -\frac{1}{2}du$

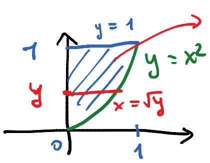
$$= \frac{1}{2}\int_0^1 u^{1/2}\, du = \frac{1}{2} \frac{u^{3/2}}{3/2}\Big|_0^1 = \frac{1}{3}$$

EXAMPLE 7. *Evaluate the following iterated integral by reversing the order of integration:*

$$I = \int_0^1 \int_{x^2}^1 x^3 \sin(y^3) \, dy dx$$

What is the region of integration D?

$$D = \{(x,y) \mid 0 \leq x \leq 1, \ x^2 \leq y \leq 1\} = \{(x,y) : 0 \leq y \leq 1, \ 0 \leq x \leq \sqrt{y}\}$$



$y = x^2 \iff x = \sqrt{y}$

$$I = \iint_D x^3 \sin(y^3)\, dA = \int_0^1\left(\int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\right) dy =$$

$$= \int_0^1 \sin(y^3)\left(\int_0^{\sqrt{y}} x^3\, dx\right) dy = \int_0^1 \sin(y^3) \frac{x^4}{4}\Big|_{x=0}^{\sqrt{y}} dy =$$

$$= \int_0^1 \sin(y^3)\left(\frac{\overbrace{(\sqrt{y})^4}^{y^2}}{4} - 0\right) dy = \frac{1}{4}\int_0^1 y^2 \sin(y^3)\, dy = \frac{1}{4}\cdot\frac{1}{3}\int_0^1 \sin u \, du =$$

u-substitution : $u = y^3$: $y: 0 \to 1$, $u: 0 \to 1$

$du = 3y^2 dy \Rightarrow y^2 dy = \frac{1}{3} du$

$$= \frac{1}{12} \, -\cos u \Big|_{u=0}^{1} = \frac{1}{12}(1 - \cos 1).$$