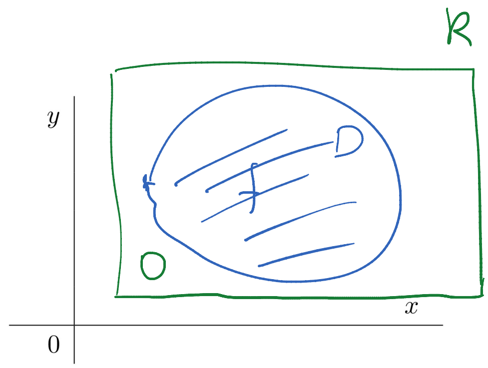
F19_LN_15_2

Sunday, October 20, 2019 9:25 PM

F19_LN_1...

15.2: Double Integrals over General Regions

`All functions below are continuous on their domains.`



Let D be a bounded region enclosed in a rectangular region R such that its boundary ∂D is sufficiently nice, for example, is a piecewise differentiable curve. We define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D. \end{cases}$$

If F is integrable over R, then we say F is *integrable* over D and we define **the double integral of f over D** by

$$\iint_D f(x,y)\,\mathrm{d}A \overset{\text{def}}{=} \iint_R F(x,y)\,\mathrm{d}A$$

FACT: If $f(x,y) \geq 0$ and f is continuous on the region D then the volume V of the solid S that lies above D and under the graph of f, i.e.

$$S = \left\{(x,y,z) \in \mathbb{R}^3 |\ 0 \leq z \leq f(x,y), (x,y) \in D\right\},$$

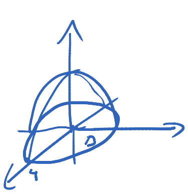
is

$$V = \iint_D f(x,y)\,\mathrm{d}A.$$

EXAMPLE 1. *Evaluate the integral*

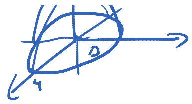
$$\iint_D \overbrace{\sqrt{16 - x^2 - y^2}}^{f(x,y)}\,\mathrm{d}A$$

where $D = \left\{(x,y) \in \mathbb{R}^2 |\ x^2 + y^2 \leq 16\right\}$ by identifying it as a volume of a solid.



$z = f(x,y) = \sqrt{16-x^2-y^2} \overset{\text{square}}{\Leftrightarrow} \underbrace{x^2+y^2+z^2 = 16 \;\&\; z \geq 0}_{\text{a hemisphere of radius 1}}$

$\iint \sqrt{16-x^2-y^2}\,dA = \frac{1}{2}$ volume of ... $= \frac{1}{2} \cdot \frac{4}{3}\pi \cdot 4^3 = \frac{128}{3}\pi$



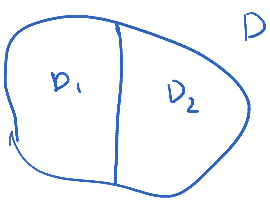
a hemisphere of radius 4

$$\iint_D \sqrt{16-x^2-y^2}\, dA = \frac{1}{2} \text{ volume of the ball of radius } 4 = \frac{1}{2}\,\frac{4}{3}\,\pi\cdot 4^3 = \frac{128}{3}\pi$$

Properties of double integrals:

- If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps their boundaries then

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D_1} f(x,y)\,\mathrm{d}A + \iint_{D_2} f(x,y)\,\mathrm{d}A.$$



- If α and β are real numbers then

$$\iint_D (\alpha f(x,y) + \beta g(x,y))\,\mathrm{d}A = \alpha \iint_D f(x,y)\,\mathrm{d}A + \beta \iint_D g(x,y)\,\mathrm{d}A.$$

- If we integrate the constant function $f(x,y) = 1$ over D, we get **area** of D:

$$\iint_D 1\,\mathrm{d}A = \underbrace{A(D)}_{\text{the area of } D}.$$

EXAMPLE 2. *If* $D = \{\underbrace{(x,y)|\ x^2+y^2 \le 25}_{\text{a disk of radius } 5}\}$ *then*

$$\iint \mathrm{d}A = \text{area of } D = \pi 5^2 = 25\pi$$

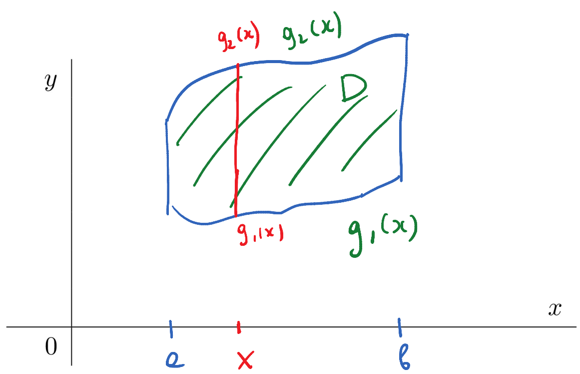
EXAMPLE 2. *If* $D = \left\{(x, y) |\ x^2 + y^2 \le 25\right\}$ *then*

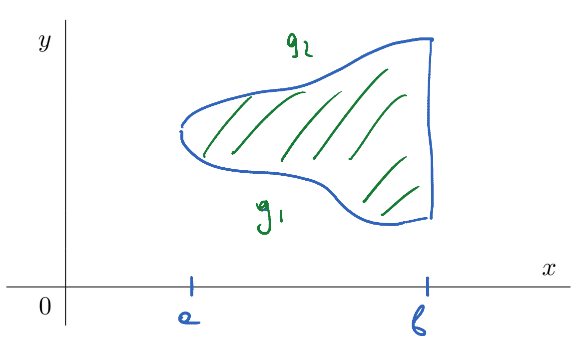
$$\iint_D \mathrm{d}A = \text{area of } D = \pi 5^2 = 25\pi$$

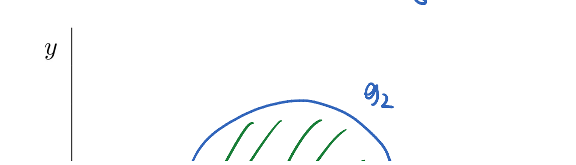
Computation of double integral:

A plain region of **TYPE I**:

$$D = \{(x, y) |\ a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

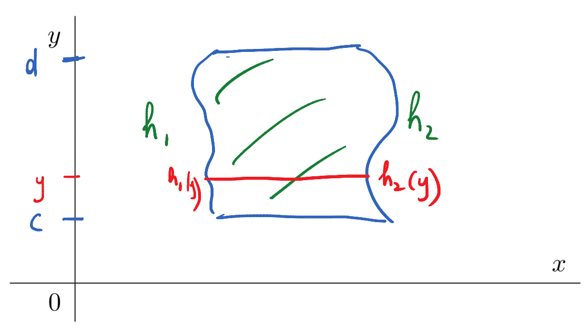


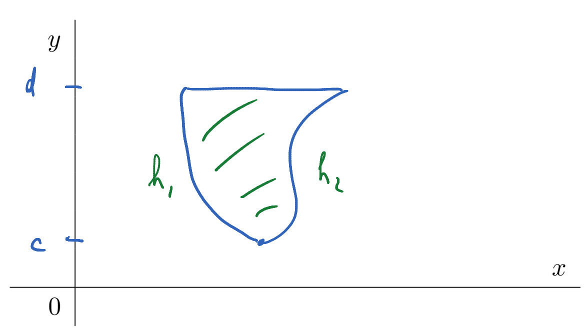


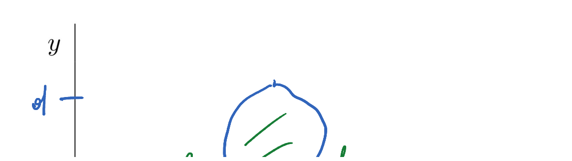


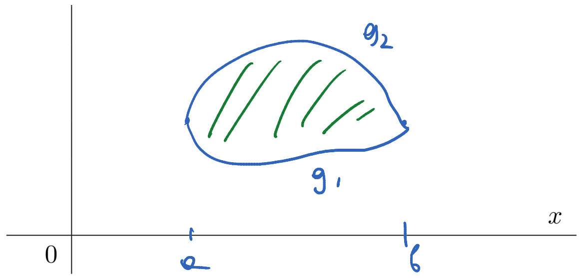
A plain region of **TYPE II**:

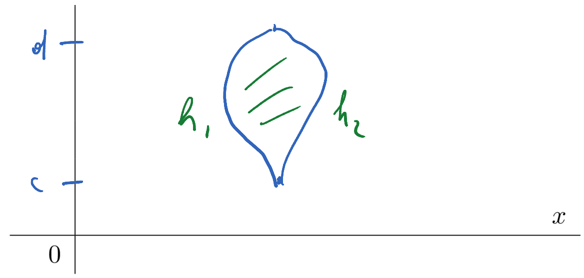
$$D = \{(x, y) |\ c \le y \le d, h_1(y) \le x \le h_2(y)\}.$$











THEOREM 3. *If D is a region of type I such that $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$ then*

Fubini Theorem

$$\iint_D f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\mathrm{d}x.$$

$g_2(x) \to$ upper curve

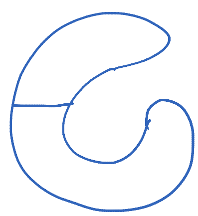
$g_1(x) \to$ lower curve

THEOREM 4. *If D is a region of type II s.t. $D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$ then*

$$\iint_D f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

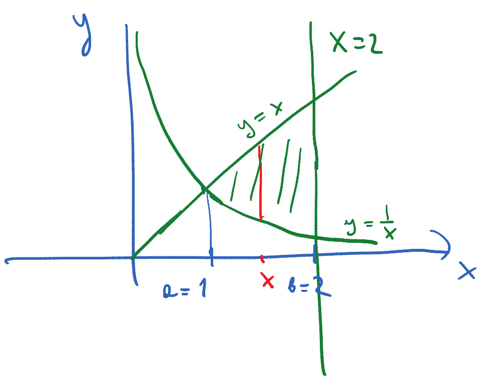
$h_2(y) \to$ right curve

$h_1(y) \to$ left curve



EXAMPLE 5. *Evaluate $I = \iint_D 30x^2y\,\mathrm{d}A$, where D is the region bounded by the lines $x = 2, y = x$ and the hyperbola $xy = 1$ in two different ways (i.e. considering D as a type I and then as a type II region).*

$xy = 1$: hyperbola



1) D as a region of type I

lower curve is $y = \frac{1}{x}$

upper curve is $y = x$

$b = 2$

$a = 1$ (for this we need to find x-component of the point of intersection of 2 graphs: $\begin{cases} y = x \\ y = \frac{1}{x} \end{cases} \Rightarrow x = \frac{1}{x} \Rightarrow x^2 = 1$, $x > 0 \to x = 1$)

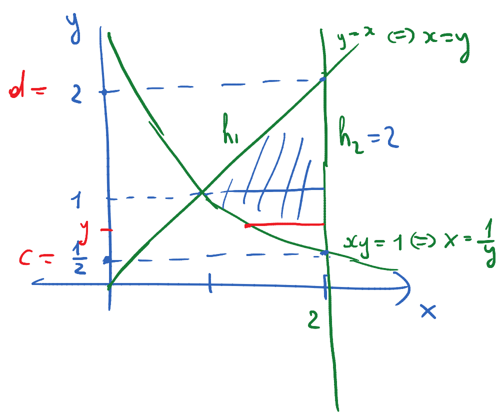
$$I = \int_1^2 \left(\int_{\frac{1}{x}}^{x} 30x^2 y\,dy \right) dx = 30 \int_1^2 x^2 \int_{\frac{1}{x}}^{x} y\,dy =$$

$$= 30 \int_1^2 x^2 \left(\frac{y^2}{2} \Big|_{\frac{1}{x}}^{x} \right) dx = \frac{30}{2} \int_1^2 x^2 \left(x^2 - \frac{1}{x^2} \right) dx = 15 \int_1^2 (x^4 - 1)\,dx =$$

$$= 30 \int_1^2 x^2 \left(\frac{y^2}{2} \Big|_{y=\frac{1}{x}}^{x} \right) dx = \underbrace{\frac{30}{2}}_{15} \int_1^2 x^2 \left(x^2 - \frac{1}{x^2}\right) dx = 15 \int_1^2 (x^4 - 1)\, dx =$$

$$= 15 \left(\frac{x^5}{5} - x\right)\Big|_{x=1}^{2} = 15\left(\frac{32}{5} - 2 - \left(\frac{1}{5} - 1\right)\right) = 15\left(\frac{31}{5} - 1\right) = \cancel{15}^{\,3} \cdot \frac{26}{\cancel{5}} = \boxed{78}$$

2) D as a region of type II



y-component of intersection of $y=\frac{1}{x}$ with $x=2$ is: $\begin{cases} y=\frac{1}{x}\\ x=2\end{cases} \Rightarrow y = \frac{1}{2}$

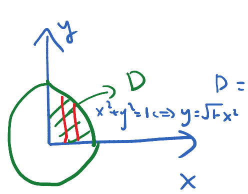
y-component of intersection of $y=x$ with $x=2$ is: $\begin{cases} y=x\\ x=2\end{cases} \Rightarrow y = 2$

y-component of intersection of $y=x$ with $y=\frac{1}{x}$ in the first quadrant is: $\begin{cases} y=x\\ y=\frac{1}{x}\end{cases} \Rightarrow x=\frac{1}{x} \Rightarrow x^2=1\ (\&\ x>0) \Rightarrow x=1 \Rightarrow y=1$

The left curve is given by $h_1(y) = \begin{cases} \frac{1}{y}, & \text{if } \frac{1}{2} \le y \le 1\\ y; & \text{if } 1\le y\le 2\end{cases}$ $\quad h_2(y)=2$

$$I = \int_{\frac{1}{2}}^{1} \left(\int_{\frac{1}{y}}^{2} 30x^2y\,dx\right)dy + \int_1^2\left(\int_y^2 30x^2 y\,dx\right)dy = \text{exercise} \ldots = \boxed{78}$$

EXAMPLE 6. *Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes* ~~$x = 0$~~, $y = z$, $\underbrace{z = 0}_{xy\text{-plane}}$ *in the first octant.*

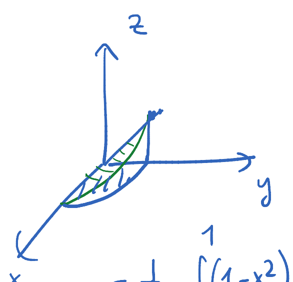


$D = \left\{(x,y): \begin{array}{l} x^2+y^2\le 1\\ x\ge 0\\ y\ge 0\end{array}\right\}$

$y = z \iff z = f(x,y) = y$

$$V = \iint_D y\, dA = \int_0^1 \int_0^{\sqrt{1-x^2}} y\, dy\, dx =$$

D as region of type 1: $D = \left\{(x,y): \begin{array}{l} 0\le x\le 1\\ 0\le y\le \sqrt{1-x^2}\end{array}\right\}$

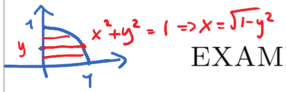


$$= \int_0^1 \frac{y^2}{2}\Big|_{y=0}^{\sqrt{1-x^2}} dx = \frac{1}{2}\int_0^1 \Big(\underbrace{\left(\sqrt{1-x^2}\right)^2}_{1-x^2} - 0\Big)\, dx =$$

$$= \frac{1}{2}\int_0^1 (1-x^2)\, dx = \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Big|_{x=0}^{1} = \frac{1}{2}\left(1-\frac{1}{3}\right) = \boxed{\frac{1}{3}}$$

$= \frac{1}{2}\int_0^1 (1-x^2)\,dx = \frac{1}{2}\left. x - \frac{x^3}{3}\right|_{x=0}^{1} = \frac{1}{2}\left(1-\frac{1}{3}\right) = \boxed{\frac{1}{3}}$

Way 2 : D as a region of type 2 : $D = \{(x,y) : 0 \le y \le 1,\ 0 \le x \le \sqrt{1-y^2}\} \Rightarrow V = \int_0^1 \int_0^{\sqrt{1-y^2}} y\,dx\,dy =$



$= \int_0^1 y\left(\int_0^{\sqrt{1-y^2}} dx\right) dy = \int_0^1 y\sqrt{1-y^2}\,dy = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 u^{1/2}\,du = \frac{1}{2}\left.\frac{u^{3/2}}{3/2}\right|_{u=0}^{1} = \frac{1}{2}\cdot\frac{1}{\frac{3}{2}} = \boxed{\frac{1}{3}}$

$u = 1-y^2$, $du = -2y\,dy \rightarrow y\,dy = -\frac{1}{2}\,du$; $y: 0 \to 1$, $u: 1 \to 0$

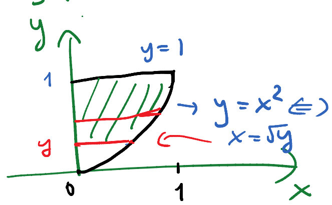
EXAMPLE 7. *Evaluate the following iterated integral by reversing the order of integration:*

$$I = \int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\,\mathrm{d}y\mathrm{d}x$$

1. Identify the region D of integration:

$D = \{(x,y) : 0 \le x \le 1,\ x^2 \le y \le 1\}$

Sketch



$D = \{(x,y) : 0 \le y \le 1,\ 0 \le x \le \sqrt{y}\}$

as region of type 2

$I = \iint_D x^3 \sin(y^2)\,dA = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\,dx\,dy =$

$= \int_0^1 \sin(y^3)\left(\int_0^{\sqrt{y}} x^3\,dx\right) dy = \int_0^1 \sin(y^3)\left(\left.\frac{x^4}{4}\right|_{x=0}^{\sqrt{y}}\right) dy = \int_0^1 \sin(y^3)\left(\frac{\overbrace{(\sqrt{y})^4}^{y^2}}{4} - 0\right) dy$

$= \frac{1}{4}\int_0^1 y^2 \sin(y^3)\,dy = \frac{1}{4}\cdot\frac{1}{3}\int_0^1 \sin u\,du = \frac{1}{12}(-\cos u)\Big|_0^1 = \frac{1}{12}(1-\cos 1)$

$y: 0 \to 1$
$u: 0 \to 1$

u- substitution: $u = y^3$:
$du = 3y^2\,dy \Rightarrow y^2\,dy = \frac{1}{3}\,du$