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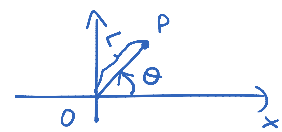
F19_LN_1...

15.3: Double Integrals in Polar Coordinates



The **polar coordinate system** consists of:

- the **pole** (or origin) labeled O;
- the **polar axis** which is a ray starting at O (usually drawn horizontally to the right);

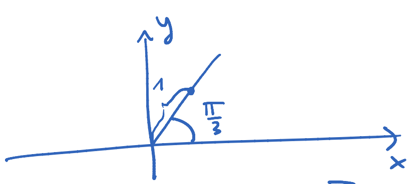
The **polar coordinates** (r, θ) of a point P:

- θ is the angle between the ~~polar~~ x- axis and the line OP (the angle is positive if measured in counter-clockwise direction from the ~~polar~~ x- axis);
- r is the distance from O to P.

EXAMPLE 1. *Plot the points whose polar coordinates are given:*

(a) $(1, \pi/3)$ **(b)** $(5, -\pi/2)$.

$r = 1$, $\theta = \frac{\pi}{3}$

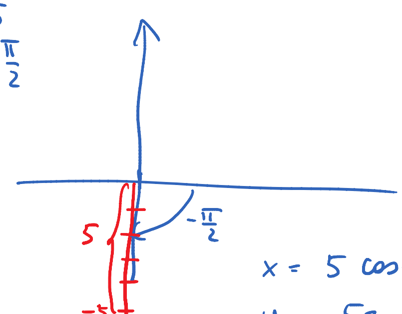


$x = 1 \cos\frac{\pi}{3} = \frac{1}{2}$

$y = 1 \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$r = 5$

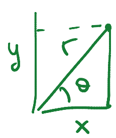
$\theta = -\frac{\pi}{2}$



$x = 5\cos\left(\frac{\pi}{2}\right) = 0$

$y = 5\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{1} = -5$

The connection between polar and Cartesian coordinates:



$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$ $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

$x = r\cos\theta$ $y = r\sin\theta$

$r^2 = x^2+y^2$ ← Pitagoras $\tan\theta = \frac{y}{x}$

REMARK 2. In converting from the Cartesian to polar coordinates we must choose θ so that the point (r, θ) lies in the correct quadrant.

EXAMPLE 3. *What curve is represented by the following polar equation*

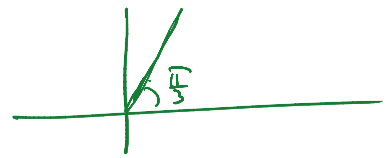
(a) $r = 12$ *(b)* $\theta = \frac{\pi}{3}$

↓ a circle of radius 12

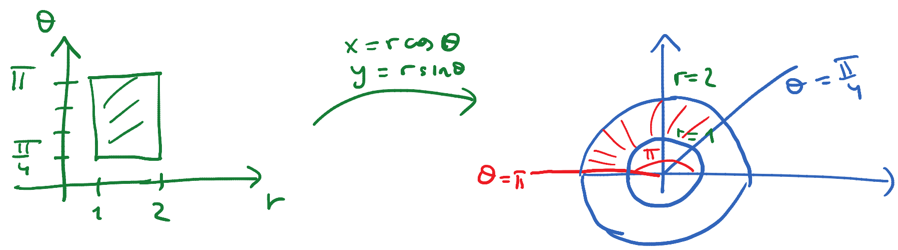
↓ The ray having angle $\frac{\pi}{3}$ with a circle of radius 12

$\sqrt{x^2+y^2}=12 \iff x^2+y^2=12^2$

The ray having angle $\frac{\pi}{3}$ with x- axis



EXAMPLE 4. *Sketch the region in the Cartesian plane consisting of points whose polar coordinates satisfy the following conditions:* $1 \le r \le 2, \quad \pi/4 \le \theta \le \pi$.



EXAMPLE 5. *Find a polar equation for the curve represented by the given Cartesian equation:*

(a) $x^2+y^2=2by$, $b>0$

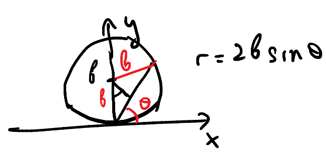
$x = r\cos\theta$
$y = r\sin\theta$

$x^2+y^2=r^2$

$r^{\cancel{2}} = 2b\cancel{r}\sin\theta$

$\boxed{r = 2b\sin\theta}$

$r \geq 0 \Rightarrow \sin\theta \geq 0 \Rightarrow 0 \le \theta \le \pi$



In cartesian coordinates

$x^2+y^2=2by \iff$

$x^2+y^2-2by=0$

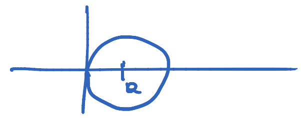
completing squares

$x^2+\underbrace{y^2-2by+b^2}_{(y-b)^2}=b^2$

$x^2+(y-b)^2=b^2 \to$

a circle of radius b centered at $(0,b)$

(b) $(x-a)^2+y^2=a^2$, $a>0$ $\longrightarrow$ circle of radius a centered at $(a,0)$



$x^2-2ax+\cancel{a^2}+y^2=\cancel{a^2}$

$x^2+y^2=2ax$

$x=r\cos\theta$

$y=r\sin\theta$

$r^2=2a\,r\cos\theta \iff r=2a\cos\theta$

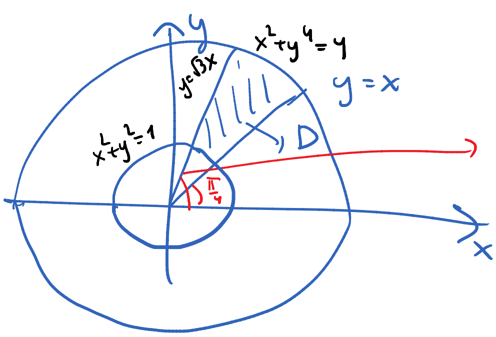
$r\geq 0 \Rightarrow \cos\theta\geq 0 \Rightarrow -\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$

Using polar coordinates to evaluate double integrals

EXAMPLE 6. *Evaluate*

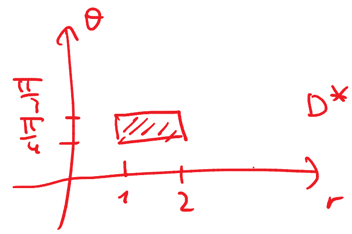
$$I=\iint_D \arctan\frac{y}{x}\,\mathrm{d}A$$

where $D=\left\{(x,y)|\ 1\leq x^2+y^2\leq 4, x\leq y\leq\sqrt{3}x, x\geq 0\right\}$.



$\arctan\sqrt{3}=\frac{\pi}{3}$

In (r,θ)-plane this is a rectangle



THEOREM 7. *Change to polar coordinates in a double integral: Let f be a continuous on the region D. Denote by D^* the region representing D in the polar coordinates (r,θ). Then*

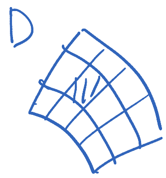
$$\iint_D f(x,y)\,\mathrm{d}A=\iint_{D^*} f(r\cos\theta,r\sin\theta)\,\overbrace{r\,\mathrm{d}r\mathrm{d}\theta}^{dA}.$$

$x=r\cos\theta$

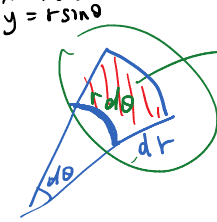
$y=r\sin\theta$

D

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,\mathrm{d}r\mathrm{d}\theta.$$



$x = r\cos\theta$
$y = r\sin\theta$



$dA = r\,dr\,d\theta$

$\|$

$$\frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2} r^2 d\theta =$$
$$= \frac{1}{2}\left(\cancel{r^2} + 2r\,dr + dr^2 - \cancel{r^2}\right) d\theta =$$
$$= r\,dr\,d\theta + \cancel{\frac{1}{2} dr^2 d\theta} \approx r\,dr\,d\theta$$

REMARK 8. Be careful not to forget the additional factor r on the right side of the formula.

Solution of Example 6:

Evaluate $I = \iint_D \arctan\frac{y}{x}\,\mathrm{d}A$, where $D = \left\{(x,y) \mid 1 \le x^2+y^2 \le 4, x \le y \le \sqrt{3}x, x \ge 0\right\}$.

Find the region D^* in (r,θ)-plane corresponding to D

$$D = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ x \le y \le \sqrt{3}x,\ x \ge 0\}$$

Plug $\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}$ into inequalities defining D

$$1 \le \underbrace{x^2+y^2}_{r^2} \le 4 \iff 1 \le r^2 \le 4 \iff 1 \le r \le 2$$

$$x \le y \le \sqrt{3}x \iff 1 \le \underbrace{\frac{y}{x}}_{tg\theta} \le \sqrt{3} \iff 1 \le tg\theta \le \sqrt{3} \iff \underbrace{\arctan 1}_{\frac{\pi}{4}} \le \theta \le \underbrace{\arctan\sqrt{3}}_{\frac{\pi}{3}}$$

$x \ge 0$

Plug polar

and we are in the 1st or 4th quadren

$$D^* = \left\{(r,\theta) : 1 \le r \le 2, \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\right\}$$ – a rectangle in (r,θ)-plane

$$I = \iint_D \arctan\frac{y}{x}\,dA \overset{\text{change in polar}}{=} \int_{D^*} \underbrace{\left(\arctan\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}\right)}_{\arctan(\tan\theta) = \theta} \underbrace{r}_{\text{a factor of change of the area element}}\,dr\,d\theta = \int_1^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} r\theta\,d\theta\,dr =$$

EXAMPLE 9. *Find the volume of the solid that lies under the paraboloid*

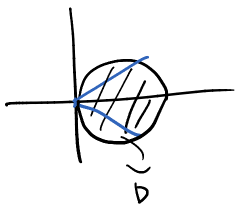
$$I = \iint_D \arctan\frac{y}{x}\, dA \underset{\text{change in polar}}{=} \int_{D^*} \Big(\underbrace{\arctan\frac{r\sin\theta}{r\cos\theta}}_{\arctan(\tan\theta)=\theta}\Big) \underbrace{r}_{\text{a factor of change of the area element}}\, dr\, d\theta = \int_1^2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} r\theta\, d\theta\, dr =$$

$$= \int_1^2 r\,dr \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \theta\, d\theta = \frac{r^2}{2}\Big|_{r=1}^{2}\ \frac{\theta^2}{2}\Big|_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{4}(4-1)\cdot\left(\frac{\pi^2}{9}-\frac{\pi^2}{16}\right) = \dots = \frac{7}{192}\pi^2$$

EXAMPLE 9. *Find the volume of the solid that lies under the paraboloid $z = \underbrace{x^2 + y^2}_{f(x,y)}$, above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$.*

Example 5(b)

In xy-plane $\quad x^2+y^2 = 2x \iff (x-1)^2 + y^2 = 1 \to$ a circle of radius 1 around $(1,0)$



$$V = \iint_D (x^2+y^2)\, dA$$

Find D^*: $D = \{(x,y) : x^2+y^2 \le 2x\} \xrightarrow[\substack{x = r\cos\theta \\ y = r\sin\theta}]{\text{Plug}}$

$x^2+y^2 \le 2x \iff$

$r^2 \le 2r\cos\theta \overset{r\ge 0}{\iff}$

$0 \le r \le 2\cos\theta$

Since $r \ge 0 \Rightarrow \cos\theta \ge 0 \iff -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$D^* = \left\{(r,\theta) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta\right\} \to \text{a region of type 1 in } (r,\theta)\text{-plane}$$

$$V = \iint_D \underbrace{(x^2+y^2)}_{r^2} dA \underset{\text{change to polar}}{=} \iint_{D^*} r^2 \cdot \underbrace{r}_{\text{the factor of change of the area element}}\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} r^3\, dr\right) d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_{r=0}^{2\cos\theta} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{16\cos^4\theta}{4}\, d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\cos^4\theta}_{(\cos^2\theta)^2}\, d\theta =$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \Rightarrow \cos^4\theta = \frac{(1+\cos 2\theta)^2}{4} = \frac{1}{4}\Big(1 + 2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\frac{1+\cos 4\theta}{2}}\Big) = \frac{1}{4}\left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)$$

$$= \cancel{4}\cdot\frac{1}{\cancel{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) = \frac{3}{2}\pi$$

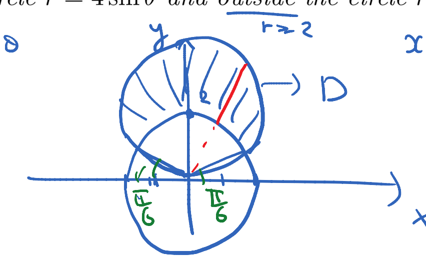
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 4\theta\, d\theta = 0$$

EXAMPLE 10. *Find the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.*

$\iff r^2 = 4r\sin\theta \iff x^2+y^2 = 4y \iff x^2 + (y-2)^2 = 4$ (see example 5a)

$r \le 4\sin\theta \qquad r \ge 2$

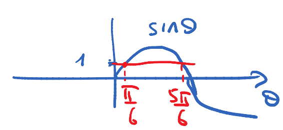
$$A = \iint_D \underset{f\equiv 1}{d}A = \iint_{D^*} r\, dr\, d\theta$$



Describe D^*:

$2 \le r \le 4\sin\theta \underset{\text{bounds for }\theta}{\Longrightarrow} 2 \le 4\sin\theta \Rightarrow \sin\theta \ge \frac{1}{2}$

Recall that $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$



Therefore $\sin\theta \ge \frac{1}{2} \iff \underline{\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}}$

$$D^* = \left\{(r,\theta) : \frac{\pi}{6} \le \theta \le \frac{5\pi}{6},\ 2 \le r \le 4\sin\theta\right\}$$

$$D^* = \left\{ (r,\theta) : \frac{\pi}{6} \le \theta \le \frac{5\pi}{6} \;,\; 2 \le r \le 4\sin\theta \right\}$$

$$A = \iint_{D^*} r\,dr\,d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{2}^{4\sin\theta} r\,dr\,d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left(16\sin^2\theta - 4\right) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}} - 2\right) d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(4 - 4\cos 2\theta - 2\right) d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(2 - 4\cos 2\theta\right) d\theta =$$

$$= 2\left(\underbrace{\frac{5\pi}{6} - \frac{\pi}{6}}_{\frac{2\pi}{3}}\right) - 2\left(\underbrace{\sin\frac{5\pi}{3}}_{-\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}} - \sin\frac{\pi}{3}\right) = \frac{4\pi}{3} + 4\cdot\frac{\sqrt{3}}{2} = \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$