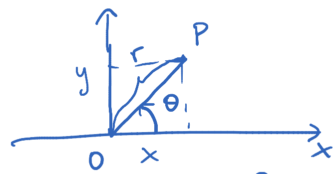
F19_LN_15_3

Sunday, October 20, 2019 9:25 PM

F19_LN_1...



$x = r\cos\theta$

$y = r\sin\theta$

15.3: Double Integrals in Polar Coordinates

The **polar coordinate system** consists of:

- the **pole** (or origin) labeled O;
- the **polar axis** (positive = x-axis) which is a ray starting at O (usually drawn horizontally to the right);

The **polar coordinates** (r, θ) (→ mod 2π) of a point P:

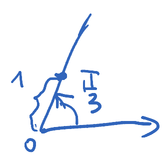
- θ is the angle between the polar axis and the line OP (the angle is positive if measured in counter-clockwise direction from the polar axis);
- r is the distance from O to P.

EXAMPLE 1. *Plot the points whose polar coordinates are given:*

(a) $(1, \pi/3)$ **(b)** $(5, -\pi/2)$.

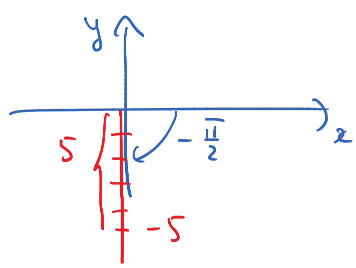
$r = 1$

$\theta = \frac{\pi}{3}$



$x = 1 \cdot \cos\frac{\pi}{3} = \frac{1}{2}$

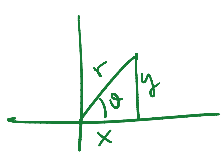
$y = 1 \cdot \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



$x = 5\overbrace{\cos\left(-\frac{\pi}{2}\right)}^{0} = 0$

$y = 5\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1} = -5$

The connection between polar and Cartesian coordinates:

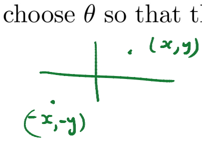


$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$ $\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$

$x = r\cos\theta$ $y = r\sin\theta$

$r^2 = x^2+y^2$ $\tan\theta = \frac{y}{x}$

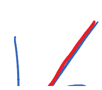
REMARK 2. In converting from the Cartesian to polar coordinates we must choose θ so that the point (r, θ) lies in the correct quadrant. For $(-x,-y)$ $\frac{-y}{-x} = \frac{y}{x}$



EXAMPLE 3. *What curve is represented by the following polar equation*

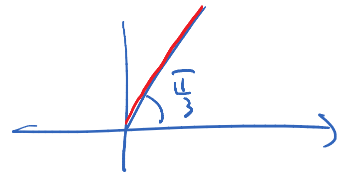
(a) $r = 12$ *(b)* $\theta = \frac{\pi}{3}$

↓ a circle of radius 12 around the origin



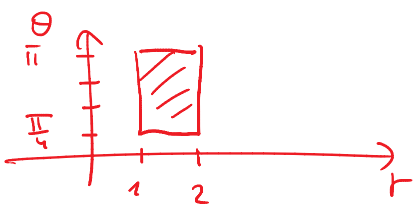
↓ a circle of radius 12 around the origin

$r = \sqrt{x^2+y^2} = 12 \iff x^2+y^2 = (12)^2$

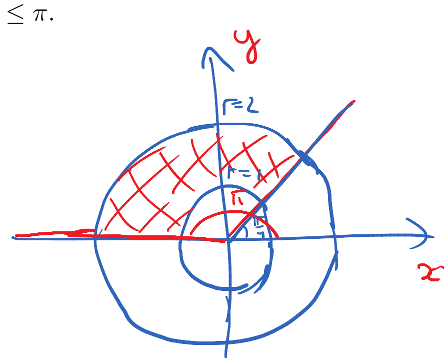


$y = \underbrace{\tan\frac{\pi}{3}}_{\sqrt{3}}\, x, \quad x \geq 0$

EXAMPLE 4. *Sketch the region in the Cartesian plane consisting of points whose polar coordinates satisfy the following conditions:* $1 \leq r \leq 2, \quad \pi/4 \leq \theta \leq \pi$.



$x = r\cos\theta$
$y = r\sin\theta$



EXAMPLE 5. *Find a polar equation for the curve represented by the given Cartesian equation:*

(a) $x^2 + y^2 = 2by$, $b > 0$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2 = r^2$

$r^{\cancel{2}} = 2b\cancel{r}\sin\theta \iff$

$\boxed{r = 2b\sin\theta}$

Since $r \geq 0 \overset{b>0}{\Longrightarrow} \sin\theta \geq 0 \Rightarrow 0 \leq \theta \leq \pi$

In Cartesian plane:

$x^2+y^2 = 2by \iff$

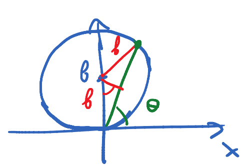
$x^2+y^2-2by = 0$

↓ complete squares

$x^2+\underbrace{y^2-2b\cdot y+b^2}_{(y-b)^2} = b^2$

$x^2+(y-b)^2 = b^2 \Rightarrow$

a circle of radius b centered at $(0, b)$



(b) $(x-a)^2+y^2=a^2$, $a>0$

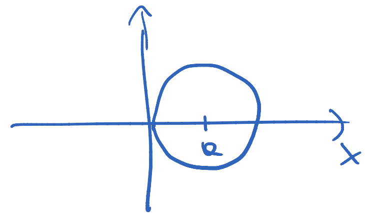
$\Updownarrow$

$x^2+y^2=2ax$

$\Updownarrow$

$\boxed{r = 2a\cos\theta} \geq 0 \Rightarrow$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

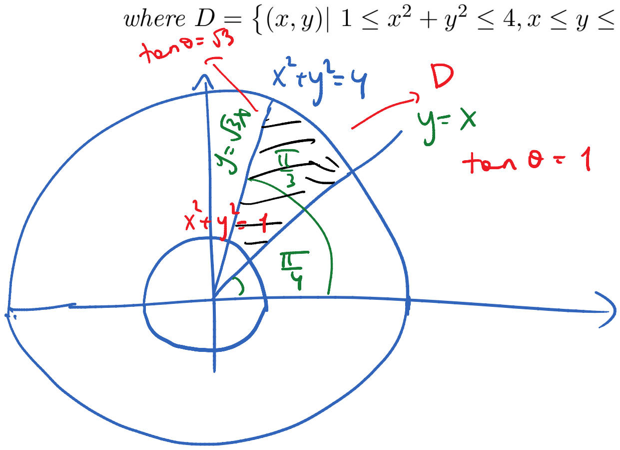


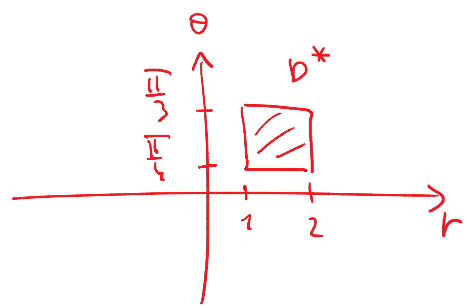
Using polar coordinates to evaluate double integrals

EXAMPLE 6. *Evaluate*

$$I = \iint_D \arctan\frac{y}{x}\,\mathrm{d}A$$

where $D = \left\{(x,y)\,|\ 1 \leq x^2+y^2 \leq 4, x \leq y \leq \sqrt{3}x, x \geq 0\right\}$.





THEOREM 7. *Change to polar coordinates in a double integral: Let f be a continuous on the region D. Denote by D^* the region representing D in the polar coordinates (r,θ). Then*

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,\underbrace{r}\,\mathrm{d}r\mathrm{d}\theta.$$

$x = r\cos\theta$
$y = r\sin\theta$

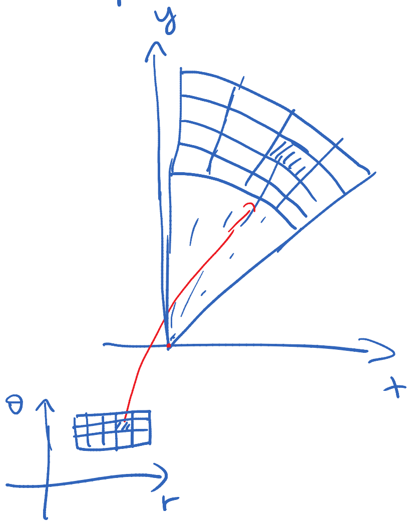
↳ additional factor of the change in the

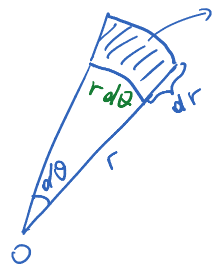
Explanation of factor r :

$x = r\cos\theta$
$y = r\sin\theta$

Explanation of factor r :

↳ additional factor of the change in the area element





$$\boxed{dA \approx r\, d\theta \cdot dr}$$

$$\|$$

$$\underbrace{\frac{1}{2}(r+dr)^2\, d\theta}_{\text{the area of a big sector}} - \underbrace{\frac{1}{2} r^2\, d\theta}_{\text{the area of a small sector}} =$$

$$= \frac{1}{2}\left(\cancel{r^2} + 2r\,dr + (dr)^2 - \cancel{r^2}\right) d\theta = r\, dr\, d\theta + \cancel{(dr)^2\, d\theta}$$

REMARK 8. Be careful not to forget the additional factor r on the right side of the formula.

Solution of Example 6:

Evaluate $I = \iint_D \arctan \frac{y}{x}\, dA$, where $D = \left\{(x,y) \mid 1 \le x^2+y^2 \le 4, x \le y \le \sqrt{3}x, x \ge 0\right\}$.

What is D^*? $D = \{(x,y) \mid 1 \le x^2 + y^2 \le 4, x \le y \le \sqrt{3}x, x \ge 0\}$

Plug $x = r\cos\theta$, $y = r\sin\theta$ to the given inequalities:

- $1 \le \underbrace{x^2+y^2}_{r^2} \le 4 \iff 1 \le r^2 \le 4 \iff \boxed{1 \le r \le 2}$
- $x \le y \le \sqrt{3}x \iff 1 \le \frac{y}{x} \le \sqrt{3} \iff 1 \le \underbrace{\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}}_{\tan\theta} \le \sqrt{3} \iff 1 \le \tan\theta \le \sqrt{3}$

  & $x \ge 0$

  ineq. do not change.

  plug polar coordinates

  and we are in 1st or 4th quadrant

$\iff \underbrace{\arctan 1}_{\frac{\pi}{4}} \le \theta \le \underbrace{\arctan\sqrt{3}}_{\frac{\pi}{3}} \iff \boxed{\frac{\pi}{4} \le \theta \le \frac{\pi}{3}} \Rightarrow D^* = \left\{(r,\theta) \;\middle|\; \begin{array}{l} 1 \le r \le 2 \\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3} \end{array}\right\}$

$$\iint_D \arctan\frac{y}{x}\, dA \underset{\text{change to polar}}{=} \iint_{D^*} \underbrace{\arctan\left(\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}\right)}_{\arctan(\tan\theta) = \theta}\, \underset{\substack{\text{a factor of the change}\\ \text{of area element}}}{r}\, dr\, d\theta = \int_1^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \theta\, r\, d\theta\, dr =$$

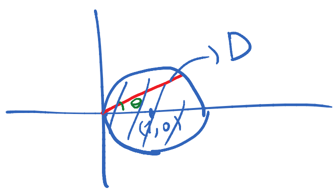
$$= \int_1^2 r\, dr \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \theta\, d\theta = \frac{1}{2} r^2 \Big|_{r=1}^{2} \cdot \frac{1}{2}\theta^2 \Big|_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{4}(4-1)\left(\frac{\pi^2}{9} - \frac{\pi^2}{16}\right) = \ldots = \frac{7}{192}\pi^2$$

EXAMPLE 9. *Find the volume of the solid that lies under the paraboloid*

$= \int \int_{\frac{\pi}{4}} \theta \, d\theta = \frac{1}{2} r^2 \Big|_{r=1} \cdot \frac{1}{2}\theta^2 \Big|_{\theta=\frac{\pi}{4}} = \frac{1}{4}(4-1)\left(\frac{\pi^2}{9}-\frac{\pi^2}{16}\right) = \dots = \frac{7}{192}\pi^2$

EXAMPLE 9. *Find the volume of the solid that lies under the paraboloid* $z = \underbrace{x^2 + y^2}_{f(x,y)}$*, above the xy-plane and inside the cylinder* $x^2 + y^2 = 2x$.

In xy-plane we have $x^2+y^2 \le 2x$ (see example 5b showing that $x^2+y^2=2x \iff (x-1)^2+y^2=1 \to$ the circle of radius 1 centered at $(1,0)$)



$$V = \iint_D (x^2+y^2)\, dA$$

What is D^* in (r,θ)-plane? $x^2+y^2 \le 2x \iff$ (plugging polar) $r^2 \le 2r\cos\theta \iff 0 \le r \le 2\cos\theta \Rightarrow \cos\theta \ge 0 \iff -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\Rightarrow D^* = \left\{(r,\theta) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta\right\} \to$ a region of type 2 in (r,θ)-plane

$$V = \iint_D \underbrace{(x^2+y^2)}_{r^2} dA \underset{\text{change to polar}}{=} \iint_{D^*} r^2 \cdot \underset{\text{factor of change of area element}}{r}\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_{r=0}^{2\cos\theta} d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 16 \cos^4\theta\, d\theta = 4 \cdot \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{4}\cos 4\theta\right) d\theta = \int_{-\pi/2}^{\pi/2} \frac{3}{2} d\theta = \frac{3}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{\frac{3}{2}\pi}$$

$\int_{-\pi/2}^{\pi/2} \cos 2\theta\, d\theta = \int_{-\pi/2}^{\pi/2} \cos 4\theta \, d\theta = 0$

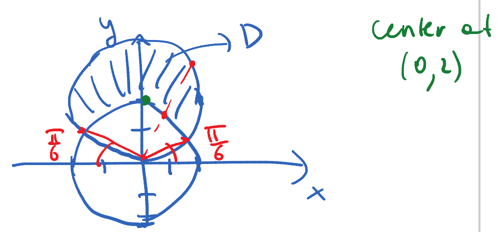
$$\cos^4\theta = (\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}})^2 = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}(1 + 2\cos 2\theta + \underbrace{\cos^2 2\theta}_{\frac{1+\cos 4\theta}{2}}) = \frac{1}{4}\left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right)$$

EXAMPLE 10. *Find the area of the region inside the circle* $r = 4\sin\theta$ *and outside the circle* $r = 2$.

see Example 5 a) $r^2 = 4r\sin\theta \iff x^2+y^2 = 4y \iff x^2+(y-2)^2 = 2^2 \to$ circle of radius 2 center at $(0,2)$

Describe D^*

$\underset{\text{outside of } r=2}{2 \le} r \underset{\text{inside of } r=4\sin\theta}{\le} 4\sin\theta \Rightarrow$

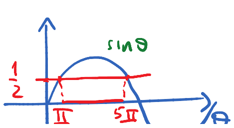


the following restriction for θ:

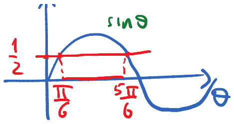
$$4\sin\theta \ge 2 \iff \sin\theta \ge \frac{1}{2}$$

Recall that $\sin\theta = \frac{1}{2}$ (if $0 \le \theta \le 2\pi$) $\iff \theta = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6} \Rightarrow \sin\theta \ge \frac{1}{2} \iff$

$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6} \Rightarrow D^* = \left\{(r,\theta) : \frac{\pi}{6} \le \theta \le \frac{5\pi}{6},\ 2 \le r \le 4\sin\theta\right\}$



$$A = \iint_D \underset{r=1}{} dA \underset{\text{change}}{=} \iint_{D^*} r\, dr\, d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_2^{4\sin\theta} r\, dr\, d\theta =$$



$$A = \iint\limits_{D} \underset{f=1}{dA} \underset{\substack{\text{change}\\ \text{to polar}}}{=} \iint\limits_{D^*} r\,dr\,d\theta = \int_{\frac{\pi}{6}}\int_{2} r\,dr\,d\theta =$$

$$= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}\left(16\sin^2\theta - 4\right)d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}} - 2\right)d\theta = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(2 - 4\cos 2\theta\right)d\theta$$

$$= 2\left(\underbrace{\frac{5\pi}{6} - \frac{\pi}{6}}_{\frac{2\pi}{3}}\right) - \cancel{4}^{\,2} \cdot \frac{\sin 2\theta}{\cancel{2}}\Bigg|_{\theta=\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{4\pi}{3} - 2\left(\underbrace{\sin\frac{5\pi}{3}}_{-\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}} - \sin\frac{\pi}{3}\right)$$

$$= \frac{4\pi}{3} - 2\cdot\cancel{2}\cdot\left(-\frac{\sqrt{3}}{\cancel{2}}\right) = \boxed{\frac{4\pi}{3} + 2\sqrt{3}}$$