F19_LN_15_4

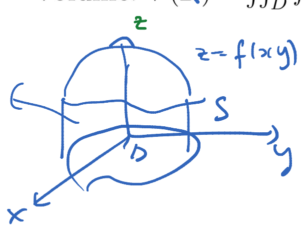
Thursday, October 24, 2019 8:36 PM

F19_LN_1...

15.4: Applications of double integral

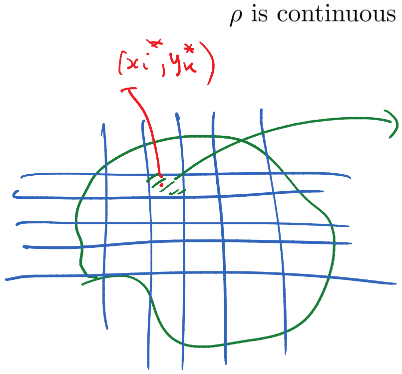
- **Area:** $A(D) = \iint_D \mathrm{d}A$

- **Volume**: $V(\overset{S}{\cancel{D}}) = \iint_D f(x,y)\,\mathrm{d}A$, where f is nonnegative on D.



- **Total Mass** m of the lamina with variable (nonhomogeneous) density $\rho(x,y)$, where the function ρ is continuous on D:

$$m = \iint_D \rho(x,y)\,\mathrm{d}A.$$



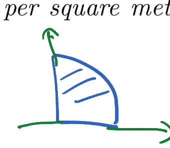
$$m_i \approx \rho(x_i^*, y_k^*)\,\Delta x_i \Delta y_k$$

$$\text{Total mass} \approx \sum_i \sum_k \rho(x_i^*, y_k^*)\,\Delta x_i \Delta y_k \underset{\substack{\Delta x_i \to 0\\ \Delta y_k \to 0}}{\Rightarrow} \iint_D \rho(x,y)\,dx\,dy$$

- **Total charge** Q: If an electric charge is distributed over a region D and the charge density (units of charge per unit area) is given by $\sigma(x,y)$ at a point (x,y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x,y)\,\mathrm{d}A.$$

EXAMPLE 1. *Charge is distributed over the part of the disk $x^2+y^2 \le 1$ in the first quadrant so that the charge density at (x,y) is $\sigma(x,y) = x^2+y^2$, measured in coulombs per square meter (C/m^2). Find the total charge.*



$$\text{Total charge} = \iint_D (x^2+y^2)\,dx\,dy \underset{\text{change to polar}}{=}$$

$$D = \{(x,y) : x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$$

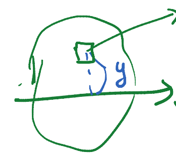
$$\underbrace{D^*}_{\text{in polar coordinates}} = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} d\theta \underbrace{\int_0^1 r^3\,dr}_{\frac{1}{4}r^4\big|_0^1} = \frac{\pi}{2}\cdot\frac{1}{4} = \boxed{\frac{\pi}{8}}$$

- **Moment** of the lamina with variable (nonhomogeneous) density $\rho(x,y)$ that occupies the region D **about the** x**-axis:**

moment about ℓ = m·r

$$M_x = \iint_D y\rho(x,y)\,\mathrm{d}A$$



$m = \rho(x,y)\,dx\,dy$

moment $= y\,\rho(x,y)\,dx\,dy$

Moment of the lamina **about the** y**-axis:**

$$M_y = \iint_D x\rho(x,y)\,\mathrm{d}A$$

- **Center of mass** , $(\bar{x},\bar{y})$, of the lamina with variable (nonhomogeneous) density $\rho(x,y)$ that occupies the region D is defined so that

- **Center of mass** , $(\bar{x}, \bar{y})$, of the lamina with variable (nonhomogeneous) density $\rho(x,y)$ that occupies the region D is defined so that

(x_2, y_2)
$\cdot m_2$

(x_1, y_1)
$\cdot m_1$

$\bar{x} = \dfrac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$

$\bar{y} = \dfrac{y_1 m_1 + y_2 m_2}{m_1 + m_2}$

$$m\bar{x} = M_y, \qquad m\bar{y} = M_x.$$

These yield

$$\bar{x} = \frac{\iint_D x\rho(x,y)\,\mathrm{d}A}{m}, \qquad \bar{y} = \frac{\iint_D y\rho(x,y)\,\mathrm{d}A}{m},$$

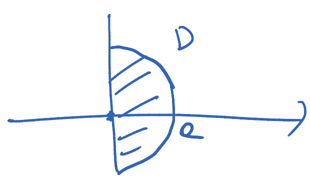
where $m = \displaystyle\iint_D \rho(x,y)\,\mathrm{d}A$.

REMARK 2. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus, the lamina balances horizontally when supported as its center of mass.

EXAMPLE 3. *Find the center of mass of the lamina that occupies the region*

$$D = \{(x,y) : x^2 + y^2 \le a^2, x \ge 0\}$$

if the density at any point is proportional to the square of its distance from the origin.



$\rho(x,y) = \underbrace{k}_{\text{constant}} \underbrace{(x^2+y^2)}_{\text{square of the distance from the origin}}$

$m = \iint_D k(x^2+y^2)\,dx\,dy = k \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^a \underbrace{r^2 \cdot r}_{r^3}\, dr\, d\theta =$

In (r,θ)-plane

$D^* = \{(r,\theta) : 0 \le r \le a, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

$= k \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^a r^3\, dr = k \cdot \pi \cdot \frac{a^4}{4}$

$\bar{x} = \dfrac{\iint x \cdot \cancel{k}(x^2+y^2)\,dx\,dy}{\cancel{k}\,\frac{\pi a^4}{4}} \underset{\text{polar change}}{=} \dfrac{4}{\pi a^4} \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^a \underbrace{r\cos\theta \cdot r^2 \cdot r}_{r^4\cos\theta}\, dr \right) d\theta =$

$= \dfrac{4}{\pi a^4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^a r^4\, dr = \dfrac{4}{\pi \cancel{a^4}} \underbrace{\sin\theta \Big|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{1-(-1)=2} \cdot \dfrac{a^{\cancel{5}}}{5} = \dfrac{8a}{5\pi}$

$\bar{y} \underset{\text{instead of } \cos\theta \text{ you have } \sin\theta}{=} \dfrac{4}{\pi a^4} \underbrace{\int_{-\frac{\pi}{2}}^{\pi/2} \sin\theta\, d\theta}_{=0} \int_0^a r^4\, dr = 0$

$\to$ also $\bar{y} = 0$ is clear from the symmetry w.r.t. the x-axis instead of $\cos\theta$ you have $\sin\theta$

Symmetry w.r.t. the x- axis