F19_LN_15_4

Thursday, October 24, 2019 8:36 PM

F19_LN_1...

15.4: Applications of double integral

- **Area:** $A(D) = \iint_D \, \mathrm{d}A$

- **Volume**: $V(\cancel{D}) = \iint_D f(x,y)\,\mathrm{d}A$, where f is nonnegative on D.

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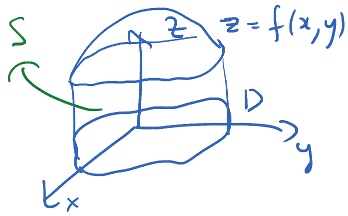


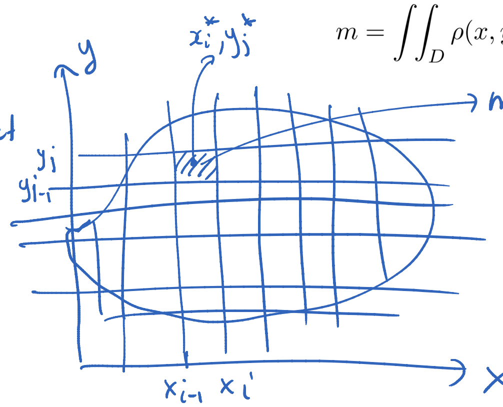
thickness ≈ 0

- **Total Mass** m of the lamina with variable (nonhomogeneous) density $\rho(x,y)$, where the function ρ is continuous on D:

per area unit

$$m = \iint_D \rho(x,y)\,\mathrm{d}A.$$

$$\rho(x,y) = \lim_{\substack{\text{as set} \\ \text{shrinks} \\ \text{to } (x,y)}} \frac{m_{\text{set around } (x,y)}}{\text{area of this set}}$$



$$m_{ij} \approx \rho(x_i^*, y_j^*)\,\underbrace{\Delta x_i \Delta y_j}_{\text{the area of the rectangle}}$$

$$\text{total mass} \approx \sum_i \sum_j \rho(x_i^*, y_j^*)\Delta x_i \Delta y_j$$

The Riemann sum

$\Delta x, \Delta y \to 0$

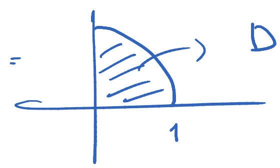
$$\iint_D \rho(x,y)\,dA$$

- **Total charge** Q: If an electric charge is distributed over a region D and the charge density (units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$Q = \iint_D \sigma(x, y)\,\mathrm{d}A.$$

EXAMPLE 1. *Charge is distributed over the part of the disk $x^2 + y^2 \le 1$ in the first quadrant so that the charge density at (x, y) is $\sigma(x, y) = x^2 + y^2$, measured in coulombs per square meter (C/m^2). Find the total charge.*

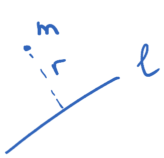
Total charge $= \iint_D (x^2+y^2)\,dA \underset{\text{polar charge}}{=} \int_0^{\frac{\pi}{2}}\int_0^1 \underbrace{r^2\cdot r}_{r^3}\,dr\,d\theta =$



$D^* = \{(r,\theta) \mid 0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2}\}$

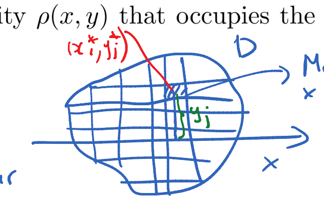
$= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^3\,dr = \frac{\pi}{2}\cdot \frac{r^4}{4}\Big|_{r=0}^{1} = \boxed{\frac{\pi}{8}}\ C$

Point mass: The moment of point mass m about l is equal to mr



- **Moment** of the lamina with variable (nonhomogeneous) density $\rho(x, y)$ that occupies the region D **about the** x**-axis:**

$$M_x = \iint_D \underset{\text{the distance to x-axis}}{y}\rho(x, y)\,\mathrm{d}A$$



Moment about x axis $= y_j\,\rho(x_i^*, y_j^*)\cdot \Delta x_i \Delta y_j$

Moment of the lamina **about the** y**-axis:**

$$M_y = \iint_D \underset{\text{the distance to y-axis}}{x}\rho(x, y)\,\mathrm{d}A$$

2 point masses

m_2

- **Center of mass** , $(\bar{x}, \bar{y})$, of the lamina with variable (nonhomogeneous) density $\rho(x, y)$ that occupies the region D is defined so that

2 point masses

m_2 (x_2,y_2)

m_1 (x_1,y_1) $\bar{x},\bar{y}$

Center of mass $(\bar{x},\bar{y})$ satisfies

$$\bar{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

$$\bar{y} = \frac{m_1y_1+m_2y_2}{m_1+m_2}$$

the distance to y-axis

- **Center of mass** , $(\bar{x},\bar{y})$, of the lamina with variable (nonhomogeneous) density $\rho(x,y)$ that occupies the region D is defined so that

$$m\bar{x} = M_y, \qquad m\bar{y} = M_x.$$

These yield

continuous analogs

$$\bar{x} = \frac{\iint_D x\rho(x,y)\,\mathrm{d}A}{m}, \qquad \bar{y} = \frac{\iint_D y\rho(x,y)\,\mathrm{d}A}{m},$$

where $m = \displaystyle\iint_D \rho(x,y)\,\mathrm{d}A$.

REMARK 2. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus, the lamina balances horizontally when supported as its center of mass.

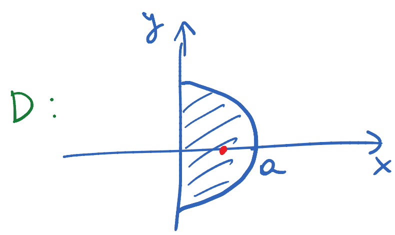
EXAMPLE 3. *Find the center of mass of the lamina that occupies the region*

$$D = \{(x,y) : x^2+y^2 \le a^2, x \ge 0\}$$

if the density at any point is proportional to the square of its distance from the origin.

$$\rho(x,y) = \underbrace{k}_{\text{constant}}\,(x^2+y^2)$$

D:



Total mass $= \iint_D k(x^2+y^2)\,dx\,dy \underset{\text{change to polar}}{=}$

In (r,θ)-plane

$$D^* = \{(r,\theta) \mid 0 \le r \le a, -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}\}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^a k\, r^2\cdot r\,dr\,d\theta = k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^a r^3\,dr = k\cdot\pi\cdot\frac{a^4}{4}$$

$$\bar{x} = \frac{\iint_D x\,\cancel{k}(x^2+y^2)\,dA}{\cancel{k}\cdot\pi\frac{a^4}{4}} = \frac{4}{\pi a^4}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^a \underbrace{r\cos\theta\cdot r^2\cdot r}_{r^4\cos\theta}\,dr\right)d\theta =$$

$$= \frac{4}{\pi a^4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta\int_0^a r^4\,dr = \frac{4}{\pi a^4}\;\sin\theta\Big|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\frac{a^5}{5} = \frac{4\cdot 2}{\pi \cancel{a^4}}\cdot\frac{a^{\cancel{5}}}{5} = \boxed{\frac{8a}{5\pi}}$$

$$= \frac{\quad}{\pi a^4} \int_{-\frac{\pi}{2}} \quad \int_0 \quad \pi a^4 \quad \underbrace{\Big|_{\theta=-\frac{\pi}{2}}}_{\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) = 1-(-1) = 2} \quad 5 \quad \pi \cancel{a^4} \quad 5 \quad \underline{15\pi}$$

$$\bar{y} \underset{\substack{\uparrow \\ \text{replace} \\ \cos\theta \text{ by } \sin\theta}}{=} \frac{4}{\pi a^4} \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta \, d\theta}_{\substack{=0 \text{ (the integral of} \\ \text{odd function} \\ \text{over the interval} \\ \text{symmetric w.r.t.} \\ \text{the origin)}}} \int_0^a r^4 \, dr = 0 \rightarrow$$

also $\bar{y} = 0$ follows from the symmetry of the lamina and its density w.r.t. the x-axis

The center of mass is at $\left(\frac{8a}{5\pi}, 0\right)$