F19_LN_15_6

Sunday, October 27, 2019 8:40 PM

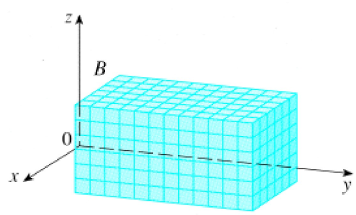
F19_LN_1...

15.6: Triple Integrals

Mass problem: *Given a solid object, that occupies the region B in $\mathbb{R}^3$, with density $\rho(x, y, z)$. Find the mass of the object.*

Solution: Let B be a rectangular box:

$$B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}$$



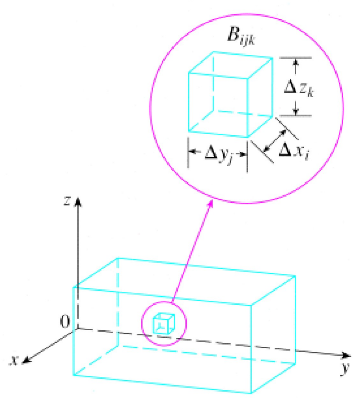
Partition in sub-boxes:

$$m_{ijk} = \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$\|P\| = \max \sqrt{\Delta x_i^2 + \Delta y_j^2 + \Delta z_k^2}$$

$$m = \lim_{\|P\| \to 0} \sum_i \sum_j \sum_k \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$m = \iiint_B \rho(x, y, z)\, \mathrm{d}V$$



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FUBINI's THEOREM: *If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ then*

$$\iiint_B f(x, y, z)\, \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y \mathrm{d}z$$

and there are 5 other possible orders in which we can integrate.

EXAMPLE 1. *Let $B = [0, 1] \times [-1, 3] \times [0, 3]$. Evaluate*

$$I = \iiint_B xye^{yz}\, \mathrm{d}V = \int_{-1}^{3}\left(\int_0^3\left(\int_0^1 x y e^{yz}\, dx\right) dz\right) dy =$$

$\iiint_B$

$$= \int_{-1}^{3}\int_{0}^{3}\left(y e^{yz}\underbrace{\int_0^1 x\,dx}_{\frac{x^2}{2}\Big|_{x=0}^{1}=\frac{1}{2}}\right)dz\,dy = \frac{1}{2}\int_{-1}^{3}\left(\int_0^3 y e^{yz}\,dz\right)dy = \frac{1}{2}\int_{-1}^{3}\cancel{y}\,\frac{e^{yz}}{\cancel{y}}\Bigg|_{z=0}^{3}dy$$

$$= \frac{1}{2}\int_{-1}^{3}\left(e^{3y}-1\right)dy = \frac{1}{2}\left(\frac{e^{3y}}{3}-y\right)\Bigg|_{y=-1}^{3} = \frac{1}{2}\left(\frac{e^{9}}{3}-3-\left(\frac{e^{-3}}{3}+1\right)\right) =$$

$$= \frac{1}{2}\left(\frac{e^{9}}{3}-\frac{e^{-3}}{3}-4\right)$$

[1]All figures are from the course textbook

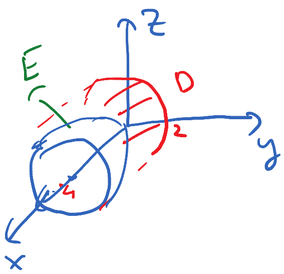
FACT: The volume of the solid E is given by the integral,

$$V = \iiint_E \mathrm{d}V.$$

FACT: The mass of the solid E with variable density $\rho(x,y,z)$ is given by the integral,

$$m = \iiint_E \rho(x,y,z)\,dV .$$

EXAMPLE 2. *Find the mass of the solid bounded by $x = y^2+z^2$ and the plane $x = 4$ if the density function is $\rho(x,y,z) = \sqrt{y^2+z^2}$.*



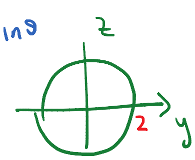
$$m = \iiint_E \sqrt{y^2+z^2}\,dV = \iint_D\left(\int_{y^2+z^2}^{4}\sqrt{y^2+z^2}\,dx\right)dy\,dz$$

D: the projection of our E to yz-plane = the interior of intersection of 2 graphs: $\begin{cases} x = y^2+z^2 \\ x = 4\end{cases} \Rightarrow y^2+z^2=4$

$$D = \{(y,z):\ y^2+z^2 \le 4\}$$

$$= \iint_{\underbrace{y^2+z^2\le 4}_{D}} \sqrt{y^2+z^2}\,\left(4-(y^2+z^2)\right)dy\,dz \underset{\text{polar change}}{=} \int_0^{2\pi}\left(\int_0^2 r\left(4-r^2\right)r\,dr\right)d\theta =$$

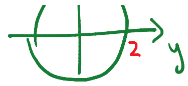
$\begin{cases} y = r\cos\theta \\ z = r\sin\theta\end{cases}$



In (r,θ)-plane:

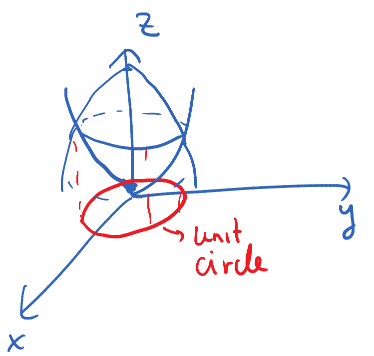
$D^* = \{(r,\theta):\ 0\le r\le 2,\ 0\le\theta\le 2\pi\}$

$$= \int_0^{2\pi} d\theta \int_0^2 \left(4r^2 - r^4\right)dr = 2\pi\left(\frac{4r^3}{3}-\frac{r^5}{5}\right)\Bigg|_{r=0}^{2} = \ldots = \frac{128}{15}\pi$$



$$= \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4)\,dr = 2\pi \left(\frac{4r^3}{3} - \frac{r^5}{5}\right)\Bigg|_{r=0}^{2} = \dots = \frac{128}{15}\pi$$

EXAMPLE 3. *Use a triple integral to find the volume of the solid bounded by the surfaces* $z = x^2 + y^2$ *and* $z = 5 - 4x^2 - 4y^2$.



$$V = \iiint_E dV = \iint_D \left(\int_{x^2+y^2}^{5-4x^2-4y^2} dz \right) dx\,dy$$

D: projection of E onto xy-plane = interior of projection of intersection of 2 graphs

To find this intersection solve the following system
$$\begin{cases} z = x^2 + y^2 \\ z = 5 - 4x^2 - 4y^2 \end{cases} \Rightarrow x^2 + y^2 = 5 - 4x^2 - 4y^2 \Leftrightarrow 5x^2 + 5y^2 = 5 \Leftrightarrow x^2 + y^2 = 1$$

$$\Rightarrow D = \{(x,y) : x^2 + y^2 \le 1\}$$

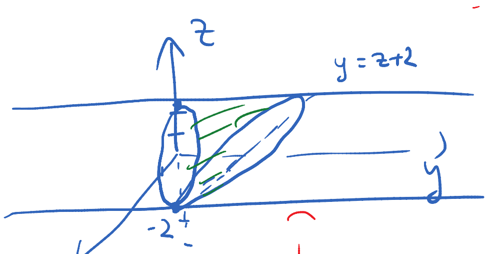
$$V = \iint_{x^2+y^2\le 1} \left(\int_{x^2+y^2}^{5-4x^2-4y^2} dz \right) dx\,dy = \iint_{x^2+y^2\le 1} \left(5 - 4x^2 - 4y^2 - (x^2+y^2)\right) dx\,dy =$$

$$= \iint_{x^2+y^2\le 1} (5 - 5x^2 - 5y^2)\,dx\,dy = 5 \iint_{x^2+y^2\le 1} (1 - x^2 - y^2)\,dx\,dy \underset{\text{polar}}{=} 5 \int_0^{2\pi} \int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= \iint\limits_{x^2+y^2\le 1} (5-5x^2-5y^2)\,dx\,dy = 5\iint\limits_{x^2+y^2\le 1}(1-x^2-y^2)\,dx\,dy \underset{\text{polar change}}{=} 5\int_0\int_0 (1-r^2)\,r\,dr\,d\theta$$

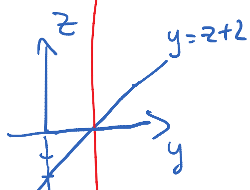
$$= 5\cdot 2\pi \int_0^1 (r-r^3)\,dr = 10\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{10\pi}{4} = \boxed{\frac{5\pi}{2}}$$

EXAMPLE 4. *Use a triple integral to find the volume of the solid bounded by the elliptic cylinder* $4x^2+z^2=4$ *and the planes* $y=0$ *and* $y=z+2$.

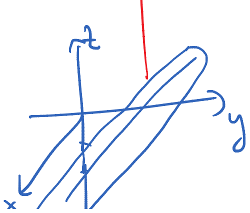


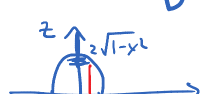
$$V = \iint\limits_{D} \int_0^{z+2} dy\; dx\,dz =$$

D — the projection of the solid to xz-plane, which in this case is the interior of the ellipse $4x^2+z^2=4$



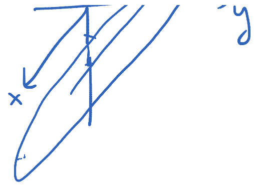
$$= \iint\limits_{D} (z+2)\,dx\,dz = \int_{-1}^{1}\left(\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} (z+2)\,dz\right)dx =$$

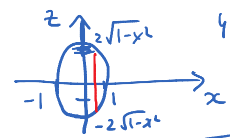




$4x^2+z^2=4$

$z = \pm 2\sqrt{1-x^2}$





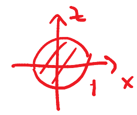
$4x^2+z^2=4$

$z=\pm 2\sqrt{1-x^2}$

$$= \int_{-1}^{1} \frac{z^2}{2}+2z \Big|_{z=-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx = \int_{-1}^{1} 0 + 2\left(2\sqrt{1-x^2}-(-2\sqrt{1-x^2})\right) dx$$

$$= 8\int_{-1}^{1} \sqrt{1-x^2}\, dx = 4 \times \left(\text{area of unit circle}\right) = 4\pi$$

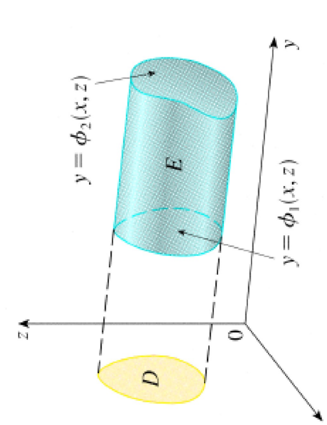
$$4\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$$



region E

A solid region of **TYPE III**:

$E=\{(x,y,z)|(x,z)\in D, \phi_1(x,z)\le y\le \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.

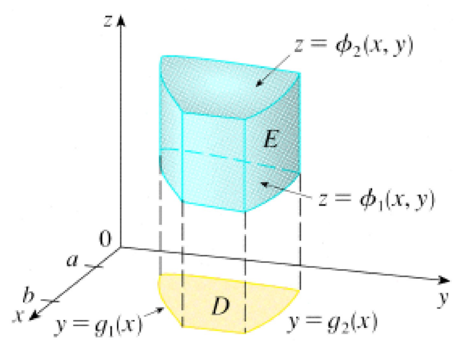
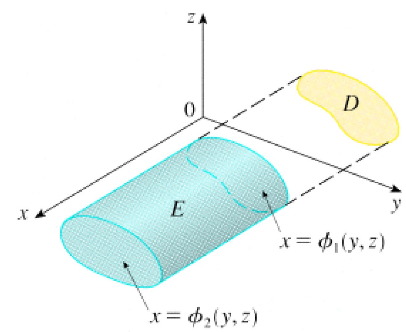


A type 3 region

$\iiint_E f(x,y,z)\,dV =$

one of its projection on the corresponding

Table 1: **Triple integrals over a general bounded region**

A solid region of **TYPE I**:	A solid region of **TYPE II**:	A solic
$E = \{(x,y,z) \| (x,y) \in D, \phi_1(x,y) \leq z \leq \phi_2(x,y)\}$ where D is the projection of E onto the xy-plane.	$E = \{(x,y,z) \| (y,z) \in D, \phi_1(y,z) \leq x \leq \phi_2(y,z)\}$ where D is the projection of E onto the yz-plane.	$E = \{(x$ where plane.
A type 1 solid region 	 A type 2 region	 A typ
$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A$	$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint \left[\int f(x,y,z)\,\mathrm{d}x\right]\mathrm{d}A$	$\iiint_E$

When we set up a triple integral it is wise to draw **two** diagrams: one of the solid region E and one of i coordinate plane.