F19_LN_15_6

Thursday, October 24, 2019 8:37 PM

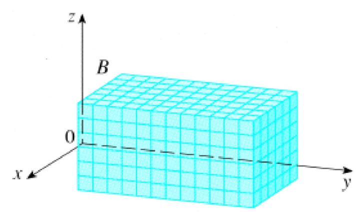
F19_LN_1...

15.6: Triple Integrals

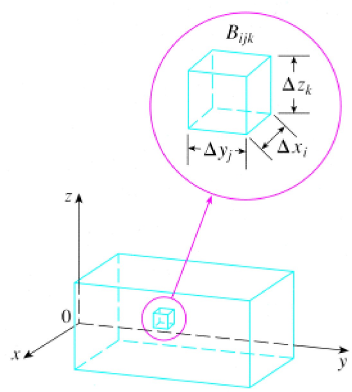
Mass problem: *Given a solid object, that occupies the region B in $\mathbb{R}^3$, with density $\rho(x, y, z)$. Find the mass of the object.*

Solution: Let B be a rectangular box:

$$B = \{(x, y, z) | a \le x \le b, c \le y \le d, r \le z \le s\}$$



Partition in sub-boxes:



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$$m_{ijk} = \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$\|P\| = \max \sqrt{\Delta x_i^2 + \Delta y_j^2 + \Delta z_k^2}$$

$$m = \lim_{\|P\| \to 0} \sum_i \sum_j \sum_k \rho(x_i^*, y_j^*, z_k^*)\Delta V_{ijk}$$

$$m = \iiint_B \rho(x, y, z)\, \mathrm{d}V$$

FUBINI's THEOREM: *If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$ then*

$$\iiint_B f(x, y, z)\, \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\mathrm{d}z$$

and there are 5 other possible orders in which we can integrate.

EXAMPLE 1. *Let $B = \underbrace{[0, 1]}_{x} \times \underbrace{[-1, 3]}_{y} \times \underbrace{[0, 3]}_{z}$. Evaluate*

$$I = \iiint_B xye^{yz}\, \mathrm{d}V = \int_{-1}^{3}\left(\int_0^3\left(\int_0^1 x y e^{yz} dx\right) dz\right) dy =$$

$$= \int_{-1}^{3}\int_0^3 y e^{yz}\left(\int_0^1 x\,dx\right) dz\,dy = \frac{1}{2}\int\left(\int y e^{yz} dz\right) dy = \frac{1}{2}\int y \frac{e^{yz}}{y}\Big|^3 dy -$$

$$= \int_{-1}^{3}\int_{0}^{3} y e^{yz} \underbrace{\left(\int_{0}^{1} x\,dx\right)}_{\frac{1}{2}} dz\,dy = \frac{1}{2}\int_{-1}^{3}\left(\int_{0}^{3} y e^{yz}\,dz\right)dy = \frac{1}{2}\int_{-1}^{3} \cancel{y}\,\frac{e^{yz}}{\cancel{y}}\Big|_{z=0}^{3} dy =$$

$$= \frac{1}{2}\int_{-1}^{3}\left(e^{3y}-1\right)dy = \frac{1}{2}\left(\frac{e^{3y}}{3}-y\right)\Big|_{-1}^{3} = \frac{1}{2}\left(\frac{e^{9}}{3}-3-\left(\frac{e^{-3}}{3}-(-1)\right)\right) =$$

$$= \frac{1}{2}\left(\frac{e^{9}}{3}-\frac{e^{-3}}{3}-4\right)$$

[1]All figures are from the course textbook

FACT: The volume of the solid E is given by the integral,

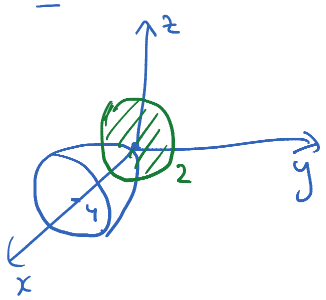
$$V = \iiint_E \mathrm{d}V.$$

FACT: The mass of the solid E with variable density $\rho(x, y, z)$ is given by the integral,

$$m = \iiint_E \rho(x,y,z)\,dV \quad .$$

EXAMPLE 2. *Find the mass of the solid bounded by $x = y^2 + z^2$ and the plane $x = 4$ if the density function is $\rho(x, y, z) = \sqrt{y^2 + z^2}$.*

circular paraboloid along x-axis



$$m = \iiint_E \sqrt{y^2+z^2}\,dV = \iint_D \left(\int_{y^2+z^2}^{4} (y^2+z^2)^{1/2}\,dx\right) dy\,dz$$

the projection of E to yz-plane = the interior of the intersection of $x = y^2+z^2$ with $x=4$: $\begin{cases} x = y^2+z^2 \\ x = 4 \end{cases} \iff \underbrace{y^2+z^2=4}_{\text{the boundary of } D}$

$$= \iint_D (y^2+z^2)^{1/2}\left(\int_{y^2+z^2}^{4} dx\right) dy\,dz =$$

$$= \iint_D (y^2+z^2)^{1/2}\left(4-(y^2+z^2)\right) dy\,dz = \int_0^{2\pi}\int_0^{2} r\,(4-r^2)\,r\,dr\,d\theta =$$

change to polar
$y = r\cos\theta$
$z = r\sin\theta$

$D = \{(y,z) : y^2+z^2 \le 4\}$

In (r,θ)-plane

$D^* = \{(r,\theta) : 0 \le r \le 2 \ , 0 \le \theta \le 2\pi\}$

$$= \int_0^{2\pi} d\theta \int_0^{2} \left(4r^2 - r^4\right) dr = 2\pi\left(\frac{4}{3}r^3 - \frac{r^5}{5}\right)\Big|_{r=0}^{2} =$$

In (r,θ)-plane

$D^* = \{(r,\theta) : 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

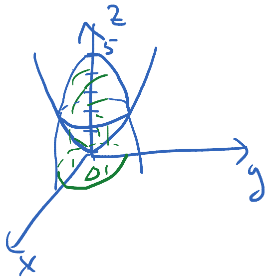
$$= \int_0^{2\pi} d\theta \int_0^{2} (4r^2 - r^4)\,dr = 2\pi\left(\frac{4}{3}r^3 - \frac{r^5}{5}\right)\Big|_{r=0}^{2}$$

$$\dots = \frac{128}{15}\pi$$

EXAMPLE 3. *Use a triple integral to find the volume of the solid bounded by the surfaces* $z = x^2 + y^2$ *and* $z = 5 - 4x^2 - 4y^2$.

internal variable of integration

and we project the solid to the xy-plane



$$V = \iiint_E dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{5-4x^2-4y^2} dz \right) dx\,dy =$$

D is the interior of the intersection of two surfaces, i.e. we have to solve the system

$$\begin{cases} z = x^2 + y^2 \\ z = 5 - 4x^2 - 4y^2 \end{cases} \Leftrightarrow x^2 + y^2 = 5 - 4x^2 - 4y^2 \Leftrightarrow 5x^2 + 5y^2 = 5 \Leftrightarrow x^2 + y^2 = 1$$

$D = \{(x,y) : x^2 + y^2 \le 1\}$

$$= \iint_D (5 - 4x^2 - 4y^2 - x^2 - y^2)\,dx\,dy = \iint_D (5 - 5x^2 - 5y^2)\,dx\,dy = 5\iint_D (1 - x^2 - y^2)\,dx\,dy$$

change

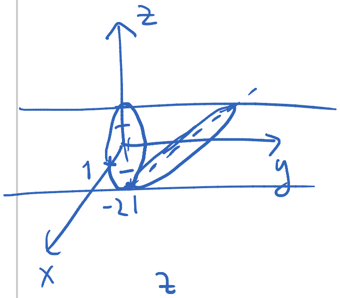
$$= 5 \int_0^{2\pi} \left(\int_0^1 \underbrace{(1 - r^2)\,r}_{r - r^3}\,dr \right) d\theta = 5 \cdot 2\pi \left(\frac{1}{2} - \frac{1}{4}\right) = 5 \cdot 2\pi \cdot \frac{1}{4} = \boxed{\frac{5\pi}{2}}$$

$$= 5\int_0 \left(\int_0 \underbrace{(1-r^2)\, r}_{r-r^3}\, dr\right) d\theta = 5\cdot 2\pi \left(\frac{1}{2}-\frac{1}{4}\right) = 5\cdot 2\pi\cdot\frac{1}{4} = \boxed{\frac{5\pi}{2}}$$

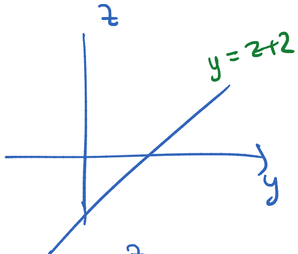
change to polar

EXAMPLE 4. *Use a triple integral to find the volume of the solid bounded by the elliptic cylinder* $\underbrace{4x^2+z^2=4}_{\text{ellipse}}$ *and the planes* $y=0$ *and* $y=z+2$.

the internal integral will be w.r.t. y



$$V = \iiint_E dV = \iint_D \left(\int_0^{z+2} dy\right) dx\, dz =$$



$$D = \{(x,z) : 4x^2+z^2 \le 4\}$$

$$= \iint_D (z+2)\, dx\, dz =$$

Another way → generalized polar

$x = \frac{r}{2}\cos\theta$

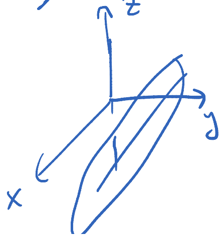
$y = r\sin\theta$

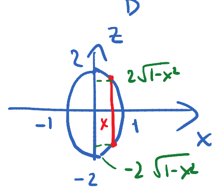
$\Downarrow$

$4x^2+y^2=r^2$

will be discussed in 15.9

$dx\,dy = \frac{1}{2} r\, dr\, d\theta$



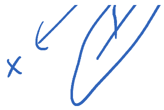


Express z in terms of x

$4x^2+z^2=4 \Rightarrow$

$z^2 = \underbrace{4-4x^2}_{4(1-x^2)} \Leftrightarrow z = \pm\sqrt{4(1-x^2)} = \pm 2\sqrt{1-x^2}$

$$= \int_{-1}^{1}\left(\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} (z+2)\, dz\right) dx = \int_{-1}^{1} \left(\frac{z^2}{2}+2z\right)\Bigg|_{z=-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx =$$

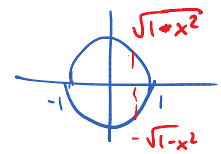


$$= \int_{-1}^{1}\left(\int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} (z+2)\,dz\right)dx = \int_{-1}^{1}\left(\frac{z^2}{2}+2z\right)\Bigg|_{z=-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} dx =$$

$$= \int_{-1}^{1} 0 + 2\left(\cdot 4\sqrt{1-x^2}\right) dx = 4\underbrace{\int_{-1}^{1} 2\sqrt{1-x^2}\,dx} = \boxed{4\pi}$$

the are of unit disk:

$$\int_{-1}^{1} 2\sqrt{1-x^2}\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \iint_{x^2+y^2\le 1} dx\,dy = \pi$$

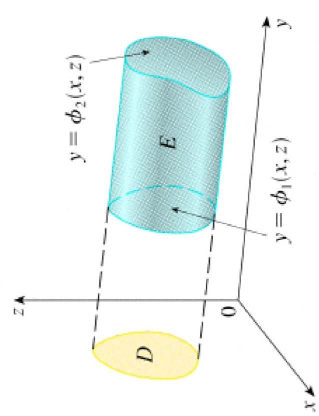


...nded region E

$z)\}$ the yz-

A solid region of **TYPE III**:

$E=\{(x,y,z)|(x,z)\in D, \phi_1(x,z)\le y\le \phi_2(x,z)\}$ where D is the projection of E onto the xz-plane.



A type 3 region

$x]dA$

$$\iiint_E f(x,y,z)\,\mathrm{d}V =$$

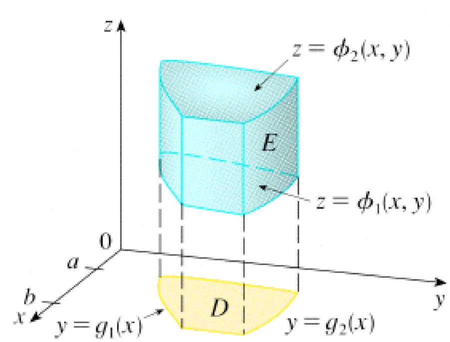
...n E and one of its projection on the corresponding

Table 1: **Triple integrals over a general bounded**

A solid region of **TYPE I**:

$E = \{(x,y,z)|(x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\}$ where D is the projection of E onto the xy-plane.

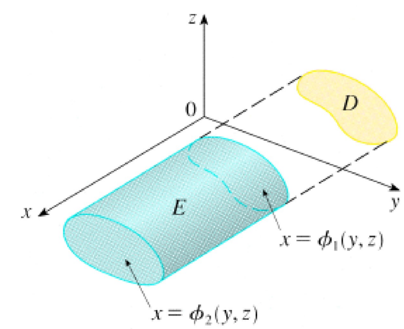
A type 1 solid region



$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A$$

A solid region of **TYPE II**:

$E = \{(x,y,z)|(y,z) \in D, \phi_1(y,z) \le x \le \phi_2(y,z)\}$ where D is the projection of E onto the yz-plane.



A type 2 region

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint \left[\int f(x,y,z)\,\mathrm{d}x\right]\mathrm{d}A$$

When we set up a triple integral it is wise to draw **two** diagrams: one of the solid region E and
coordinate plane.