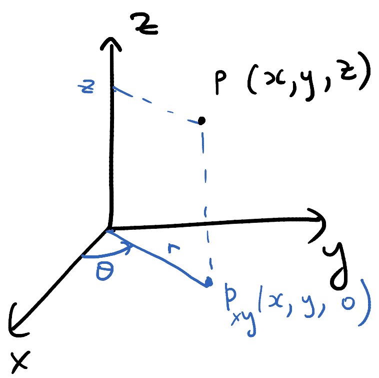
F19_LN_15_7

Tuesday, October 29, 2019 7:18 PM

F19_LN_1...

15.7: Triple integrals in cylindrical coordinates

- **Cylindrical coordinates:**



$P(x, y, z) \in \mathbb{R}^3$
In the cylindrical coordinates P is represented by the ordered triple (r, θ, z), where r, θ are the polar coordinates of P_{xy} and z is the directed distance from the xy-plane to P:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z \quad ,$$

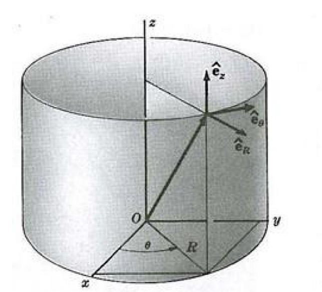
where

$$r^2 = x^2+y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

REMARK 1. The cylindrical coordinates

$$\begin{aligned} &x = r\cos\theta \\ &y = r\sin\theta \\ &z = z \\ &r \geq 0, \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

are useful in problems that involve *symmetry about the z-axis.*



(a) Cylindrical

https://i.stack.imgur.com/FgSBF.jpg

r = const
is a circular
cylinder

EXAMPLE 2. *Find an equation in cylindrical coordinates for the cone*

$$z = \sqrt{x^2+y^2} \quad \rightarrow \quad \text{upper half of a cone}$$

Plug $x = r\cos\theta$, $y = r\sin\theta$: $z = r$

THEOREM 3. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in cylindrical coordinates. Then*

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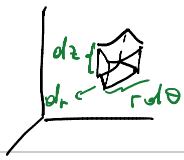
$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\,\mathrm{d}V^*, =$$

$$= \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\,\underbrace{r}_{\substack{\text{factor of}\\ \text{change of}\\ \text{volume}\\ \text{element}}}\,dr\,d\theta\,dz$$

where

$$\mathrm{d}V^* = \underline{r}\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta.$$

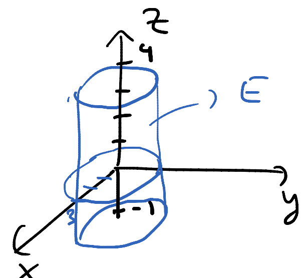
$$dV \simeq r\,d\theta\,dr\,dz$$



EXAMPLE 4. *The density at any point of the solid E,*

$$E = \left\{(x,y,z) : x^2+y^2 \le 9, -1 \le z \le 4\right\},$$

equals to its distance from the axis of E. Find the mass of E.



$E = \{(x,y,z) : x^2+y^2 \le 9, \underline{-1 \le z \le 4}\}$

$\Downarrow$

$r^2 \le 9 \iff 0 \le r \le 3$

No restriction on θ: $0 \le \theta \le 2\pi$

The corresponding solid in (r,θ,z)-space is

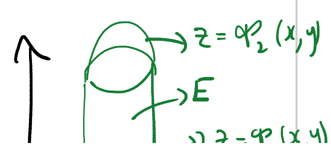
$E^* = \{(r,\theta,z) : 0 \le r \le 3, 0 \le \theta \le 2\pi, -1 \le z \le 4\}$ → a rectangular box

$\rho(x,y,z)$? the axis of E is z-axis ⇒

The distance of the point with coordinate (x,y,z) to the z-axis is $r = \sqrt{x^2+y^2}$

$$m = \iiint_E \sqrt{x^2+y^2}\,dV \underset{\substack{\text{cylindrical}\\ \text{change}}}{=} \iiint_{E^*} r\cdot r\,dr\,d\theta\,dz = \int_{-1}^{4}\left(\int_0^{2\pi}\left(\int_0^3 r^2\,dr\right)d\theta\right)dz =$$

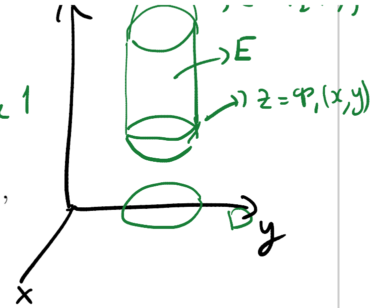
$$= \int_{-1}^{4} dz \int_0^{2\pi} d\theta \int_0^3 r^2\,dr = 5\cdot 2\pi\cdot \underbrace{\frac{3^3}{3}}_{9} = \boxed{90\pi}$$



REMARK 5. If E is a solid region of type I, i.e.

$\to$ solid of type 1

$$E = \{(x,y,z) | (x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\},$$



where D is the projection of E onto the xy-plane then, as we know,

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A.$$

Passing to cylindrical coordinates here we actually have to replace D by its image D^* in polar coordinates and $\mathrm{d}z\,\mathrm{d}A$ by $r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

EXAMPLE 6. *Find the volume of the solid E bounded by the surfaces*

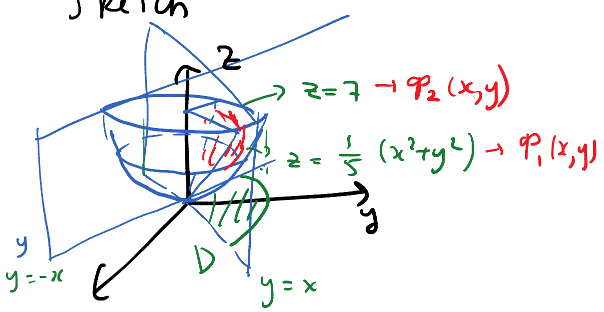
$$y = x, \quad y = -x, \quad x^2 + y^2 = 5z, \quad z = 7$$

so that $y \ge 0$.

$z = \frac{1}{5}(x^2+y^2)$

~~circular~~ paraboloid

Sketch



What is the projection of this solid on the xy-plane? = projection of the top of the solid, which is a quater of a disk.

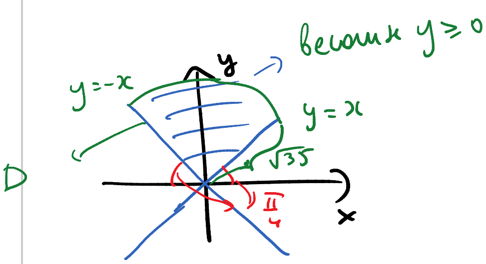
What is the radius of this disk?

The boundary of this disk is obtained by intersection of $z = \frac{1}{5}(x^2+y^2)$ with $z=7$

Solve the system

$$\begin{cases} z = \frac{1}{5}(x^2+y^2) \\ z = 7 \end{cases} \Rightarrow 7 = \frac{1}{5}(x^2+y^2) \Rightarrow x^2+y^2 = 35 \Rightarrow$$

the radius is $\sqrt{35}$



The region in (r,θ)-plane corresponding to D is

$$D^* = \{(r,\theta) : 0 \le r \le \sqrt{35}, \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$$

$$V(E) = \iint \left(\int^7 dz\right) dx\,dy \qquad \iint\left(7 - \frac{1}{5}(x^2+y^2)\right) dx\,dy =$$

$$V(E) = \iint_D \left(\int_{\frac{1}{5}(x^2+y^2)}^{7} dz\right) dx\,dy = \iint_D \left(7 - \frac{1}{5}(x^2+y^2)\right) dx\,dy \underset{\substack{\text{change}\\ \text{to polar}}}{=}$$

change to cylindrical (=) to polar

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\sqrt{35}} \underbrace{\left(7 - \frac{1}{5}r^2\right) r}_{7r - \frac{1}{5}r^3}\, dr\, d\theta = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \int_0^{\sqrt{35}} \left(7r - \frac{1}{5}r^3\right) dr = \underbrace{\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)}_{\frac{\pi}{2}} \left(\frac{7r^2}{2} - \frac{1}{20}r^4\right)\Big|_{r=0}^{\sqrt{35}}$$

$$= \frac{\pi}{2}\left(\frac{7}{2}\cdot 35 - \frac{35^2}{20}\right) = \frac{\pi}{2}\, 35 \cdot \frac{7}{2}\left(1 - \underbrace{\frac{5}{10}}_{\frac{1}{2}}\right) = \frac{\pi}{8}\cdot 35 \cdot 7 = \boxed{\frac{245\pi}{8}}$$