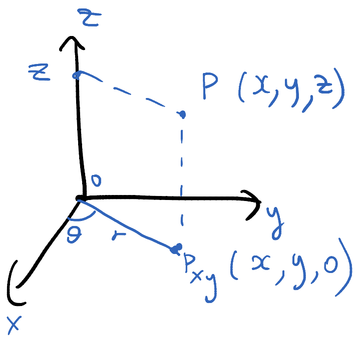
F19_LN_15_7

Tuesday, October 29, 2019 7:18 PM

F19_LN_1...

15.7: Triple integrals in cylindrical coordinates

- **Cylindrical coordinates:**



$P(x, y, z) \in \mathbb{R}^3$
In the cylindrical coordinates P is represented by the ordered triple (r, θ, z), where r, θ are the polar coordinates of P_{xy} and z is the directed distance from the xy-plane to P:

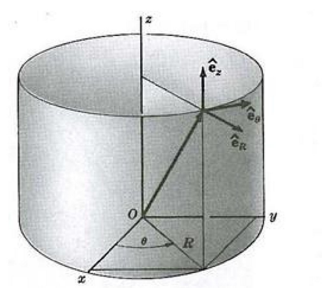
$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z ,$$

where

$$r^2 = x^2+y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

REMARK 1. The cylindrical coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$r \geq 0, \quad 0 \leq \theta \leq 2\pi$$



(a) Cylindrical

are useful in problems that involve *symmetry about the* z*-axis.*

https://i.stack.imgur.com/FgSBF.jpg

EXAMPLE 2. *Find an equation in cylindrical coordinates for the cone*

$$z = \sqrt{x^2+y^2} \rightarrow \text{upper half of the cone}$$

Plug $x = r\cos\theta$, $y = r\sin\theta$: $\boxed{z = r}$

THEOREM 3. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in cylindrical coordinates. Then*

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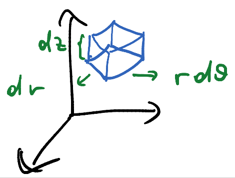
$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\,\mathrm{d}V^* = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\underbrace{r}_{\text{factor of change of element of volume}}\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

where

$$\underbrace{\mathrm{d}V^*}_{\text{the volume element in }(x,y,z)\text{-space}} = \underbrace{r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta}_{\text{the volume element in }(r,\theta,z)\text{-space}}.$$



$$dV = \underbrace{r\,d\theta\,dr}_{\text{Area of the base}}\;\underbrace{dz}_{\text{height}}$$

EXAMPLE 4. *The density at any point of the solid E,*

$$E = \left\{(x,y,z) : x^2 + y^2 \le 9, -1 \le z \le 4\right\},$$

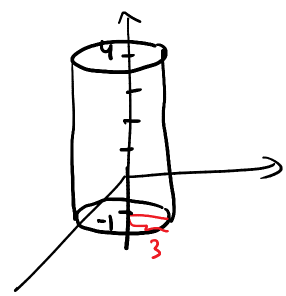
equals to its distance from the axis of symmetry *of E. Find the mass of E.*

What is the corresponding solid E^* in (r,θ,z)-space?

$x^2+y^2 \le 9 \iff r^2 \le 9 \iff 0 \le r \le 3$

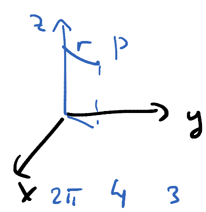
There is no restriction on θ (in addition to the initial ones) $\iff 0 \le \theta \le 2\pi$

$-1 \le z \le 4$



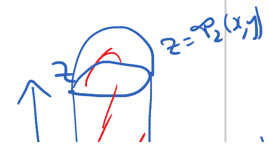
$E^* = \{(r,\theta,z) : 0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ -1 \le z \le 4\}$

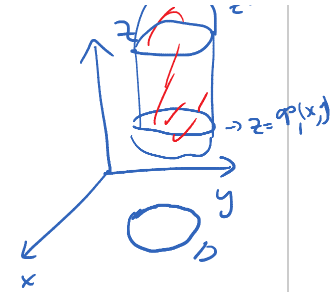
$\rho(x,y) = r = \sqrt{x^2+y^2}$



$$m(E) = \iiint_E \sqrt{x^2+y^2}\,dV \underset{\text{cylindrical change}}{=} \int_0^{2\pi}\int_{-1}^{4}\int_0^{3} \underbrace{r}_{\text{the density}} \cdot \underbrace{r}_{\text{factor of the change of the volume element}}\,dr\,dz\,d\theta =$$

$$= \int_0^{2\pi} d\theta \int_{-1}^{4} dz \int_0^{3} r^2\,dr = 2\pi \cdot 5 \cdot \underbrace{\frac{3^3}{3}}_{9} = \boxed{90\pi}$$





REMARK 5. If E is a solid region of type I, i.e. $\to$ of type 1

$$E = \{(x,y,z) | (x,y) \in D, \phi_1(x,y) \le z \le \phi_2(x,y)\},$$

where D is the projection of E onto the xy-plane then, as we know,

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A.$$

Passing to cylindrical coordinates here we actually have to replace D by its image D^* in polar coordinates and $\mathrm{d}z\,\mathrm{d}A$ by $r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

EXAMPLE 6. *Find the volume of the solid E bounded by the surfaces*

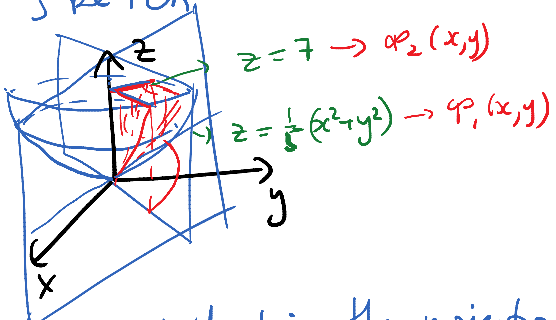
$$y = x, \quad y = -x, \quad x^2 + y^2 = 5z, \quad z = 7$$

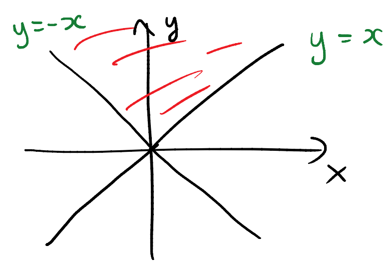
so that $y \ge 0$.

$z = \frac{1}{5}(x^2+y^2)$

circular paraboloid

Sketch





What is the projection of E on (x,y)-plane?

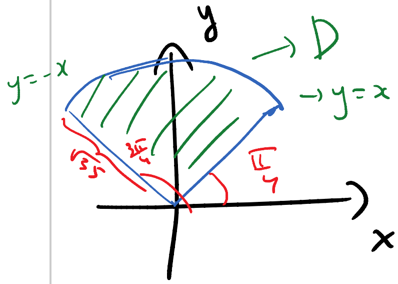
The projection is a sector of a disk (actually a quater of a disk)

What is the radius of this disk? The boundary of the disk is in the intersection of $z = \frac{1}{5}(x^2+y^2)$ and $z = 7$. So we have to solve the system

$$\begin{cases} z = \frac{1}{5}(x^2+y^2) \\ z = 7 \end{cases} \Rightarrow 7 = \frac{1}{5}(x^2+y^2) \Leftrightarrow x^2+y^2 = 35$$

The radius is $\sqrt{35}$

$z = 7$



The radius is $\sqrt{35}$

The corresponding region D^* in (r,θ)-plane is

$$D^* = \left\{ (r,\theta) \;:\; 0 \le r \le \sqrt{35} \,,\; \frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \right\}$$

$$V(E) = \iint_D \left(\int_{\frac{1}{5}(x^2+y^2)}^{7} dz \right) dx\,dy = \iint_D \left(7 - \frac{1}{5}(x^2+y^2) \right) dx\,dy =$$

change to polar

change to cylindrical $\Longleftrightarrow$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^{\sqrt{35}} \left(7 - \frac{1}{5} r^2 \right) \cdot r \, dr \, d\theta = \left(\frac{3\pi}{4} - \frac{\pi}{4} \right) \int_0^{\sqrt{35}} \left(7r - \frac{r^3}{5} \right) dr =$$

$$= \frac{\pi}{2} \left(7\frac{r^2}{2} - \frac{r^4}{20} \right) \Bigg|_{r=0}^{\sqrt{35}} = \frac{\pi}{2} \left(7 \cdot \frac{35}{2} - \frac{35^2}{20} \right) = \frac{\pi}{2} \, \frac{7 \cdot 35}{2} \left(\underbrace{1 - \frac{5}{10}}_{\frac{1}{2}} \right) =$$

$$= \frac{\pi}{8} \cdot 245 = \boxed{\frac{245\pi}{8}}$$