F19_LN_15_8

Thursday, October 31, 2019 6:20 PM

F19_LN_1...

15.8: Triple integrals in spherical coordinates

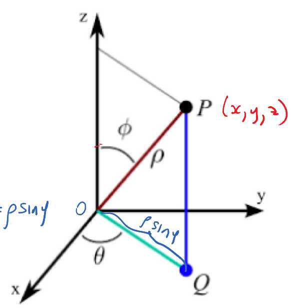
• **Spherical coordinates** of P is the ordered triple (ρ, θ, ϕ) where $|OP| = \rho$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

$x = \overbrace{\rho \sin\varphi}^{r} \cos\theta$

$y = \underbrace{\rho \sin\varphi}_{r} \sin\theta$

$z = \rho \cos\varphi$

$|OQ| = r = \rho \sin\varphi$



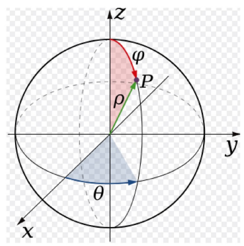
The spherical coordinates

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$
$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

are especially useful in problems where there is *symmetry about the origin.*

Note that

$x^2+y^2+z^2 = \rho^2$



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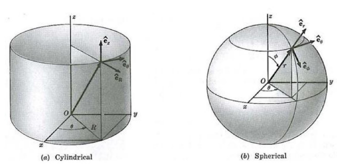

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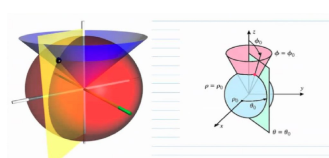


https://www.youtube.com/watch?v=Q-RUZIboBeE

EXAMPLE 1. *Find equation in spherical coordinates for the following surfaces.*

(a) $x^2 + y^2 + z^2 = 16 \iff \rho^2 = 16 \overset{\rho \geq 0}{\iff} \boxed{\rho = 4}$

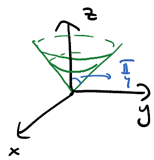
$r \to$ from cylindrical

$x = \underbrace{\rho \sin\varphi}_{r} \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

(b) $z = \sqrt{x^2 + y^2}$



$z = \rho \cos\varphi$

$x^2 + y^2 = r^2 = \rho^2 \sin^2\varphi \Rightarrow \sqrt{x^2+y^2} = \rho \sin\varphi$

$0 \leq \varphi \leq \pi \Rightarrow \sin\varphi \geq 0$

$z = \sqrt{x^2+y^2} \iff \rho\cos\varphi = \rho\sin\varphi \Rightarrow \tan\varphi = 1 \Rightarrow \boxed{\varphi = \frac{\pi}{4}}$

(c) $z = \sqrt{3x^2 + 3y^2} = \sqrt{3}\sqrt{x^2+y^2}$

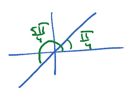
Plug the expressions of x, y & z in spherical coordinates

$\rho\cos\varphi = \sqrt{3}\rho\sin\varphi \Rightarrow \cos\varphi = \sqrt{3}\sin\varphi \Rightarrow \tan\varphi = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\varphi = \frac{\pi}{6}}$

(d) $x = y$

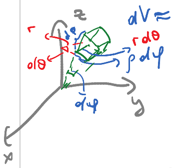
plug spherical coordinates

$\cancel{\rho\sin\varphi}\cos\theta = \cancel{\rho\sin\varphi}\sin\theta \iff \tan\theta = 1 \overset{0 \leq \theta \leq 2\pi}{\Rightarrow} \theta = \frac{\pi}{4}$ or $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$



•Triple integrals in spherical coordinates

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho\cos\phi$$
$$\rho \ge 0, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$



$dV \approx \rho\, d\varphi\, r\, d\theta\, d\rho = \rho\, r\, d\rho\, d\theta\, d\varphi = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$

$$(x,y,z) \xrightarrow{\text{cylindrical}} (r,\theta,z) \xrightarrow{\text{spherical}} (\rho,\theta,\phi)$$

$$x = r\cos\theta \qquad z = \rho\cos\phi$$
$$y = r\sin\theta \qquad r = \rho\sin\phi$$
$$z = z \qquad \theta = \theta$$

$dz\, dr\, d\theta = \rho\, d\rho\, d\varphi\, d\theta$

$(z,r,\theta) \to (\rho,\varphi,\theta)$ is cylindrical change

$$dV = dx\, dy\, dz = r\, \underbrace{d\theta\, dz\, dr}_{\rho\, d\rho\, d\varphi\, d\theta} = \underbrace{r}_{\rho\sin\varphi}\, \rho\, d\rho\, d\varphi\, d\theta = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

Geom. explanation

THEOREM 2. *Let $f(x,y,z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in spherical coordinates. Then*

$$\iiint_E f(x,y,z)\, dV = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

the factor of the change of the volume element

EXAMPLE 3. *Evaluate* $I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\, dV$ *where* $E = \{(x,y,z) : 9 \le x^2+y^2+z^2 \le 16\}$

$9 \le \underbrace{x^2+y^2+z^2}_{\rho^2} \le 16 \iff 9 \le \rho^2 \le 16 \iff 3 \le \rho \le 4$

No additional restriction for θ & φ $\Rightarrow$

The corresponding solid in (ρ,θ,φ)-space is

$E^* = \{(\rho,\theta,\varphi) : 3 \le \rho \le 4,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi\}$ → a rectangular box

$$\iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\, dV = \iiint_{E^*} \underbrace{e^{\sqrt{(\rho^2)^3}}}_{e^{\rho^3}}\, \underbrace{\rho^2\sin\varphi}_{\text{the factor of change of Volume element}}\, d\rho\, d\varphi\, d\theta =$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \int_0^{\pi} \sin\varphi\, d\varphi \int_3^4 \rho^2 e^{\rho^3}\, d\rho = 2\pi\, (-\cos\varphi)\Big|_{\varphi=0}^{\pi}\ \frac{1}{3}\int_{27}^{64} e^u\, du =$$

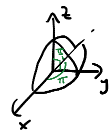
product of 3 integrals

$u = \rho^3$
$du = 3\rho^2 d\rho \Rightarrow$
$\rho^2 d\rho = \frac{1}{3} du$

$\rho: 3 \to 4$
$u: 3^3 \to 4^3$

$$= 2\pi\, \underbrace{(-\cos\pi - (-\cos 0))}_{2}\ \frac{1}{3}(e^{64} - e^{27}) = \boxed{\frac{4\pi}{3}(e^{64} - e^{27})}$$

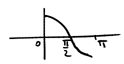
EXAMPLE 4. *Write the integral* $\iiint_E f(x,y,z)\,\mathrm{d}V$ *in spherical coordinates where*



(a) $E = \{(x,y,z) : x^2+y^2+z^2 \le 1, y \ge 0, z \ge 0\}$.

$0 \le x^2+y^2+z^2 \le 1 \iff 0 \le \rho \le 1$

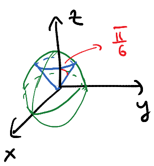
$z \ge 0 \iff \rho\cos\varphi \ge 0 \iff \cos\varphi \ge 0 \iff 0 \le \varphi \le \frac{\pi}{2}$



$y \ge 0 \iff \underbrace{\rho}_{\ge 0}\underbrace{\sin\varphi}_{\ge 0}\sin\theta \ge 0 \iff \sin\theta \ge 0 \iff 0 \le \theta \le \pi$

$$\iiint_E f(x,y,z)\,dV = \int_0^{\pi}\int_0^{\pi/2}\int_0^{1} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

(b) *E is the ice cream cone-shaped solid, which is cut from the sphere of radius* 5 *by the cone* $\phi = \pi/6$.



$0 \le \rho \le 5$

$0 \le \varphi \le \frac{\pi}{6}$

For θ there is no additional restrictions

so $0 \le \theta \le 2\pi$

$E^* = \{(\rho,\theta,\varphi) : 0 \le \rho \le 5,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{6}\}$ rectangular box

$$\iiint_E f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{5} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

EXAMPLE 5. *Evaluate the integral by changing to spherical coordinates:*

$$I = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \overbrace{(x^2+y^2+z^2)}^{\rho^2}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

What is E (over which solid we integrate?)

$E = \{(x,y,z) : \sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2},\ 0 \le x \le \sqrt{9-y^2},\ 0 \le y \le 3\}$

$z = \sqrt{x^2+y^2}$ the upper cone with angle $\frac{\pi}{4}$ with z-axis

$\rho\cos\varphi = \rho\sin\varphi$ plug spherical

$\tan\varphi = 1 \Rightarrow \varphi = \frac{\pi}{4}$

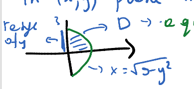
$z = \sqrt{18-x^2-y^2}$ square

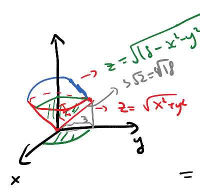
$x^2+y^2+z^2 = 18$ & $z \ge 0$

an upper hemisphere of radius $\sqrt{18} = 3\sqrt{2}$

$x = \sqrt{9-y^2} \iff x^2+y^2 = 9,\ x \ge 0$

In (x,y)-plane the projection D of E is $D = \{(x,y) : 0 \le x \le \sqrt{9-y^2}, 0 \le y \le 3\}$





E^*?

$E^* = \{(\rho,\theta,\varphi) : 0 \le \rho \le \sqrt{18},\ 0 \le \varphi \le \frac{\pi}{4},\ 0 \le \theta \le \frac{\pi}{2}\}$

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sqrt{18}} \rho^2 \cdot \underbrace{\rho^2\sin\varphi}_{\text{Jacobian}}\,d\rho\,d\varphi\,d\theta$$

$$= \underbrace{\int_0^{\pi/2} d\theta}_{\frac{\pi}{2}} \underbrace{\int_0^{\pi/4} \sin\varphi\,d\varphi}_{-\cos\varphi\Big|_{\varphi=0}^{\pi/4} = -\frac{1}{\sqrt{2}}+1} \underbrace{\int_0^{\sqrt{18}=3\sqrt{2}} \rho^4\,d\rho}_{\frac{\rho^5}{5}\Big|_{\rho=0}^{3\sqrt{2}}} =$$

$$= \frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{5}\cdot \underbrace{3^5}_{243}\underbrace{(\sqrt{2})^5}_{4\sqrt{2}} = \dots = \frac{\pi}{10}\left(1-\frac{1}{\sqrt{2}}\right)972\sqrt{2}$$