F19_LN_15_8

Thursday, October 31, 2019 6:21 PM

F19_LN_1...

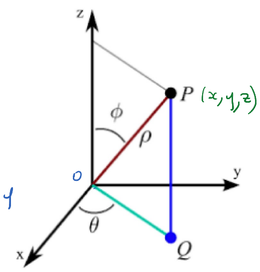
15.8: Triple integrals in spherical coordinates

• **Spherical coordinates** of P is the ordered triple (ρ, θ, ϕ) where $|OP| = \rho$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$.

$$\begin{cases} x = \underbrace{\rho \sin\phi}_{r} \cos\theta \\ y = \underbrace{\rho \sin\phi}_{r} \sin\theta \\ z = \rho \cos\phi \end{cases}$$

$|OQ| = r = \rho \sin\phi$

r ↓ from polar



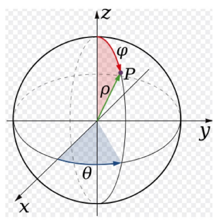
The spherical coordinates

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$
$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$

are especially useful in problems where there is *symmetry about the origin.*

Note that

$x^2+y^2+z^2 = \rho^2$



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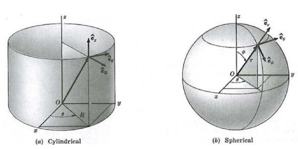

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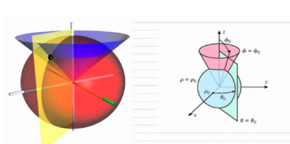


https://www.youtube.com/watch?v=Q-RUZIboBeE

EXAMPLE 1. *Find equation in spherical coordinates for the following surfaces.*

(a) $x^2 + y^2 + z^2 = 16$ $\iff$ $\rho^2 = 16 \iff \rho = 4$ (remind that $\rho \geq 0$)

(b) $z = \sqrt{x^2 + y^2}$

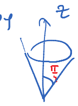
$x^2 + y^2 = \rho^2 \sin^2\varphi$ ($\rho > 0$, $\sin\varphi > 0$)

Plug $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$

$\rho \cos\varphi = \rho \sin\varphi$

$\Updownarrow$

$\tan\varphi = 1 \Rightarrow \boxed{\varphi = \frac{\pi}{4}}$



(c) $z = \sqrt{3x^2 + 3y^2} = \sqrt{3}\sqrt{x^2+y^2}$

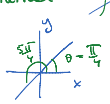
$x^2 + y^2 = \rho^2 \sin^2\varphi$

Plug the spherical coordinates

$\rho \cos\varphi = \sqrt{3}\rho \sin\varphi \iff \tan\varphi = \frac{1}{\sqrt{3}} \iff \boxed{\varphi = \frac{\pi}{6}}$

(d) $x = y$ → plane

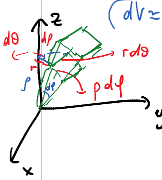
Plug spherical $\rho \sin\varphi \cos\theta = \rho \sin\varphi \sin\theta \iff \tan\theta = 1 \iff \boxed{\theta = \frac{\pi}{4} \text{ or } \frac{\pi}{4} + \pi = \frac{5\pi}{4}}$



•Triple integrals in spherical coordinates

Geometric explanation

$dV = d\rho\,\rho\,d\phi \cdot r\,d\theta = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ (with $r = \rho\sin\phi$)



$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho\cos\phi$$
$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi$$

$(x, y, z) \xrightarrow{\text{cylindrical}} (r, \theta, z) \rightarrow (\rho, \theta, \phi)$

$x = r\cos\theta$ $\quad z = \rho\cos\phi$

$y = r\sin\theta$ $\quad r = \rho\sin\phi$

$z = z$ $\quad \theta = \theta$ → cylindrical type

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz = r\rho\,d\rho\,d\theta\,d\phi = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ (with $dr\,d\theta\,dz = \rho\,d\rho\,d\theta\,d\phi$, $r = \rho\sin\phi$)

THEOREM 2. *Let $f(x, y, z)$ be a continuous function over a solid $E \subset \mathbb{R}^3$. Let E^* be its image in spherical coordinates. Then*

$$\iiint_E f(x,y,z)\,dV = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

($\rho^2\sin\phi$ → the factor of change of the volume element)

EXAMPLE 3. *Evaluate* $I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\,dV$ *where* $E = \{(x, y, z) : 9 \leq x^2 + y^2 + z^2 \leq 16\}$.

(→ a region between two spheres)

Find the corresponding E^*

$9 \leq \underbrace{x^2+y^2+z^2}_{\rho^2} \leq 16 \iff 9 \leq \rho^2 \leq 16 \iff 3 \leq \rho \leq 4$

θ & φ does not restriction in addition to the original one, i.e.

$0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi$

$E^* = \{(\rho, \theta, \varphi) : 3 \leq \rho \leq 4, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \varphi \leq \pi\}$

$I = \iiint_E e^{\sqrt{(x^2+y^2+z^2)^3}}\,dV = \iiint_{E^*} e^{\sqrt{(\rho^2)^3}}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$ (spherical change; $e^{\sqrt{(\rho^2)^3}} = e^{\rho^3}$; $\rho^2\sin\varphi$: the factor of change of the volume element)

$\int_0^{2\pi}\int_0^{\pi}\int_3^4 \rho^2 e^{\rho^3}\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\,d\varphi \int_3^4 \rho^2 e^{\rho^3}\,d\rho =$ (product of 3 single variable integrals)

$u = \rho^3 \Rightarrow$ $\rho: 3 \to 4$, $u: 3^3 \to 4^3$; $du = 3\rho^2 d\rho \Rightarrow \rho^2 d\rho = \frac{1}{3}du$

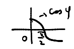
$= 2\pi\left(-\cos\varphi\Big|_{\varphi=0}^{\pi}\right)\cdot\frac{1}{3}\int_{3^3}^{4^3} e^u\,du = 2\pi\left(\underbrace{-\cos\pi - (-\cos 0)}_{1+1=2}\right)\frac{1}{3}\left(e^{64} - e^{27}\right) = \boxed{\frac{4\pi}{3}\left(e^{64} - e^{27}\right)}$

EXAMPLE 4. *Write the integral* $\iiint_E f(x, y, z)\,\mathrm{dV}$ *in spherical coordinates where*

(a) $E = \{(x, y, z) : x^2 + y^2 + z^2 \le 1, y \ge 0, z \ge 0\}$.

E^*? ① $0 \le x^2+y^2+z^2 \le 1 \iff 0 \le \rho^2 \le 1 \iff 0 \le \rho \le 1$

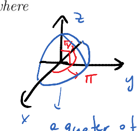
③ $z \ge 0 \iff \rho\cos\varphi \ge 0 \iff \cos\varphi \ge 0 \iff 0 \le \varphi \le \frac{\pi}{2}$



② $y \ge 0 \iff \rho\sin\varphi\sin\theta \ge 0 \iff \sin\theta \ge 0 \iff 0 \le \theta \le \pi$

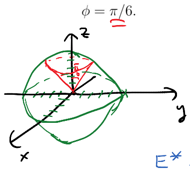
$\iiint_E f(x,y,z)\,dV = \int_0^{\pi}\int_0^{\pi/2}\int_0^{1} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

Geometrically



a quarter of a ball

(b) *E is the ice cream cone-shaped solid, which is cut from the sphere of radius 5 by the cone* $\phi = \pi/6$.



What is E^*: $0 \le \rho \le 5$, $0 \le \varphi \le \frac{\pi}{6}$

no additional restrictions for θ: $0 \le \theta \le 2\pi$

$E^* = \{(\rho, \theta, \varphi) : 0 \le \rho \le 5,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{6}\}$ → a rectangular box in (ρ, θ, φ)-space

$$\iiint_E f(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{5} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$\rho^2\sin\varphi$ — the factor of change of the volume element (Jacobian)

EXAMPLE 5. *Evaluate the integral by changing to spherical coordinates:*

$$I = \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \overbrace{(x^2 + y^2 + z^2)}^{\rho^2}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.$$

What is the solid E in (x, y, z)-space over which we integrate?

$E = \{(x, y, z) : \sqrt{x^2+y^2} \le z \le \sqrt{18-x^2-y^2},\ 0 \le x \le \sqrt{9-y^2},\ 0 \le y \le 3\}$

$z = \sqrt{x^2+y^2}$ — the upper cone with the angle $\frac{\pi}{4}$ with z-axis

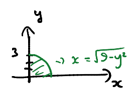
$z = \sqrt{18-x^2-y^2} \iff$ (square) $x^2+y^2+z^2 = 18$ & $z \ge 0$ — a hemisphere of radius $\sqrt{18} = 3\sqrt{2}$

Why $\varphi = \frac{\pi}{4}$? (plug spherical) $\rho\cos\varphi = \rho\sin\varphi$, $\tan\varphi = 1 \iff \varphi = \frac{\pi}{4}$

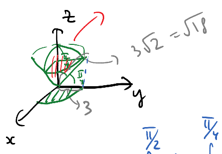
The projection D of E on the xy-plane is

$D = \{(x, y) : 0 \le y \le 3,\ 0 \le x \le \sqrt{9-y^2}\}$

$x = \sqrt{9-y^2} \iff x^2+y^2 = 9$ & $x \ge 0$ → semicircle



a quarter of the ice-cream cone



$E^* = \{(\rho, \theta, \varphi) : 0 \le \rho \le \sqrt{18} = 3\sqrt{2},\ 0 \le \varphi \le \frac{\pi}{4},\ 0 \le \theta \le \frac{\pi}{2}\}$ ← rectangular box

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3\sqrt{2}} \rho^2 \cdot \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$\rho^2\sin\varphi$ — the Jacobian; = product of 3 integrals

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/4} \sin\varphi\, d\varphi \int_0^{3\sqrt{2}} \rho^4\, d\rho = \frac{\pi}{2}\cdot(-\cos\varphi)\Big|_{\varphi=0}^{\pi/4}\cdot\frac{(3\sqrt{2})^5}{5} =$$

$$\frac{\pi}{10}\left(1-\frac{1}{\sqrt{2}}\right)3^5(\sqrt{2})^5 = \frac{\pi}{10}\left(1-\frac{1}{\sqrt{2}}\right)243\cdot 4\cdot\sqrt{2} = \boxed{\frac{\pi}{10}\left(1-\frac{1}{\sqrt{2}}\right)972\sqrt{2}}$$

$(\sqrt{2})^4\sqrt{2} = 4\sqrt{2}$