F19_LN_15_9

Sunday, November 3, 2019 10:08 PM

F19_LN_1...

15.9: Change Of Variables In Double Integral

Examples of a change of variables:

- substitution rule

after the change — the initial integral

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_\alpha^\beta f(u)\,\mathrm{d}u.$$

$u = g(x)$

$du = \underbrace{g'(x)}_{\text{the factor of change of the length element}}\,dx$

The intervals of integration: $D=(\alpha,\beta)$, $D^*=(a,b)$ s.t. $\alpha=g(a)$, $\beta=g(b)$

- conversion to polar coordinates:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\; r\,\mathrm{d}r\mathrm{d}\theta.$$

- conversion to cylindrical coordinates:

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|$$

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z)\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

- conversion to spherical coordinates:

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right|$$

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

We call the equations that define the change of variables a **transformation:**

$$x = x(u,v), \quad y = y(u,v).$$

triple compressing: $u = \frac{x}{3}$ →

double stretching: $v = 2y$ →

EXAMPLE 1. *Determine the new region that we get by applying the transformation $x = 3u$, $y = v/2$ to the region $D = \left\{(x,y) \mid \frac{x^2}{36} + y^2 \le 1\right\}$.*

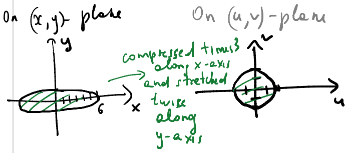
different scaling on x & y axis

→ ellipse with its interior

Plug $x = 3u$ & $y = \frac{v}{2}$ to $\frac{x^2}{36} + y^2 \le 1$: $\frac{(3u)^2}{36} + \left(\frac{v}{2}\right)^2 \le 1 \iff$

$\iff \frac{9u^2}{\cancel{36}\,4} + \frac{v^2}{4} \le 1 \iff \frac{u^2}{4} + \frac{v^2}{4} \le 1 \iff$

$u^2+v^2 \le 4$ → the disk of radius 2



DEFINITION 2. *The* **Jacobian** *of the transformation $x = x(u,v)$, $y = y(u,v)$ is*

determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

EXAMPLE 3. *Compute the Jacobian of the transformation $x = r\cos\theta$, $y = r\sin\theta$.*

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta =$$

$= r$ → factor in the change to polar

$x = x(u,v)$
$y = y(u,v)$

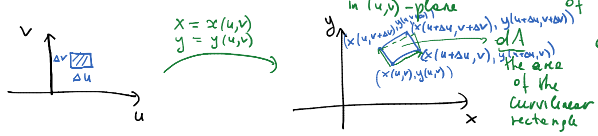
Change of variables for a double integral:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{\underset{\sim}{D^*}} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v.$$

D^* corresponds to D in (u,v)-plane

the absolute value of the Jacobian

Explanation



dA the area of the curvilinear rectangle

$dA \approx$ area of the green parallelogram

Recall that

If $\vec{a} = \langle a_1, a_2\rangle$, $\vec{b} = \langle b_1, b_2\rangle$ $\Rightarrow$ Area of parallelogram generated by $\vec{a}$ & $\vec{b}$ $= \left|\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2\end{pmatrix}\right|$

In our case $\underbrace{\Delta x \approx dx}$ $\underbrace{\Delta y \approx dy}$

$$\vec{a} = \left(x(u+\Delta u,v) - x(u,v),\ y(u+\Delta u,v) - y(u,v)\right)$$

$$\underset{\substack{\downarrow\\ \text{use approximation by differential}}}{\approx} \left(\frac{\partial x}{\partial u}\cdot\Delta u,\ \frac{\partial y}{\partial u}\cdot\Delta u\right) + \text{terms of higher order} \quad (1)$$

$$\vec{b} = \langle x(u,v+\Delta v) - x(u,v),\ y(u,v+\Delta v) - y(v)\rangle$$

$$\approx \left(\frac{\partial x}{\partial v}\Delta v,\ \frac{\partial y}{\partial v}\Delta v\right) + \text{terms of higher order} \quad (2)$$

$$dA \approx \left|\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2\end{pmatrix}\right| \underset{\substack{\downarrow\\ \text{plug }(1)\,\&\,(2)}}{\approx}$$

$$\approx \left|\det\begin{pmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v\end{pmatrix}\right| + \text{terms of higher order}$$

$$\approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta v$$

EXAMPLE 4. *Evaluate*

$$\iint_D e^{\frac{y-x}{y+x}}\,\mathrm{d}A$$

where D is triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$.

$u = y - x$
$v = y + x$

We need $\frac{\partial(x,y)}{\partial(u,v)}$

Inverse Jacobian: As a consequence of chain rule + some linear algebra:

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \Big\}\ \text{inverse Jacobian}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1\end{vmatrix} = -1\cdot 1 - 1\cdot 1 = -2$$

$\Downarrow$ Inverse Jacobian rule

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$$

Another way: express x & y in terms of u & v:

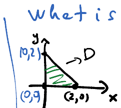
$\begin{cases} u = y - x \\ v = y + x\end{cases} \overset{\text{Linear system w.r.t. } x\,\&\,y}{\Longrightarrow}$ Eliminate x by summing up the equations

$u + v = 2y \Rightarrow y = \frac{u+v}{2} \Rightarrow x = v - y = v - \frac{u+v}{2} = \frac{v-u}{2}$

$x = \frac{v-u}{2}$, $y = \frac{u+v}{2}$ $\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$

(The previous method is more practical)

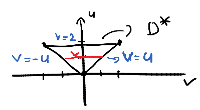
What is D^*



Since u, v depends on x, y linearly, then D^* will be also a triangle in (u,v)-plane with vertices, which are images of vertices of D

$(x,y) \xrightarrow[\substack{u=y-x\\ v=y+x}]{} (u,v)$

$(0,0) \longrightarrow (0,0)$
$(2,0) \rightarrow (-2,2)$
$(0,2) \rightarrow (2,2)$



$D^* = \{(u,v): 0 \le v \le 2,\ -v \le u \le v\}$ — as region of type 2

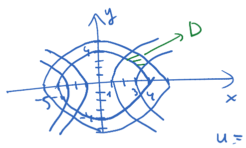
$$\iint_D e^{\frac{y-x}{y+x}}\,dA \underset{\text{change of variable}}{=} \iint_{D^*} e^{\frac{u}{v}}\cdot\underbrace{\frac{1}{2}}_{\text{absolute value of Jacobian}}\,du\,dv \underset{\substack{\text{It is more convenient}\\ \text{to integrate w.r.t. } u \text{ first}}}{=} \frac{1}{2}\int_0^2\left(\int_{-v}^{v} e^{\frac{u}{v}}\,du\right)dv =$$

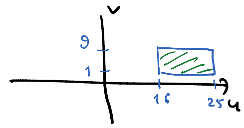
$$= \frac{1}{2}\int_0^2 \left.\frac{e^{\frac{u}{v}}}{\frac{1}{v}}\right|_{u=-v}^{u=v} = \frac{1}{2}\int_0^2 v\,(e^1 - e^{-1})\,dv = \frac{1}{2}(e^1 - e^{-1})\cdot\frac{2^2}{2} = \boxed{e - \frac{1}{e}}$$

EXAMPLE 5. *Find mass of a lamina that occupies the region*

$$D = \left\{(x,y)|16 \le \underbrace{x^2+y^2}_{u} \le 25, \quad 1 \le \underbrace{x^2-y^2}_{v} \le 9, \quad \underbrace{x \ge 0, y \ge 0}_{\text{the first quadrant}}\right\}$$

with density $\rho(x,y) = 8xy$.





$D^* = \{(u,v) : 16 \le u \le 25, \ 1 \le v \le 9\}$

a rectangle

$u = x^2+y^2$
$v = x^2-y^2$

Jacobian?

Inverse Jacobian $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -4xy - 4xy = -8xy$

Jacobian itself: $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{-8xy} \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \underset{x,y \ge 0}{=} \frac{1}{8xy}$

$m = \iint_D \underbrace{8xy}_{\rho(x,y)}\, dx\, dy \underset{\text{change of coordinates}}{=} \iint_{D^*} \cancel{8xy} \frac{1}{\cancel{8xy}}\, du\, dv = \int_{16}^{25}\int_{1}^{9} du\, dv =$

$= (25-16)(9-1) = 9 \cdot 8 = \boxed{72}$