F19_LN_15_9

Sunday, November 3, 2019 10:08 PM

F19_LN_1...

15.9: Change Of Variables In Double Integral

Examples of a change of variables:

- substitution rule

$$\underbrace{\int_a^b f(g(x))g'(x)\,\mathrm{d}x}_{\text{The resulting integral}} = \underbrace{\int_\alpha^\beta f(u)\,\mathrm{d}u}_{\text{Initial integral}}.$$

$u=g(x) \Rightarrow du = \underbrace{g'(x)}_{\text{the factor of change of length element}}dx$

$D=[\alpha,\beta]$ in u-line
$D^*=[a,b]$ in x-line, where $\alpha=g(a)$, $\beta=g(b)$

- conversion to polar coordinates:

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(r\cos\theta, r\sin\theta)\; r\,\mathrm{d}r\mathrm{d}\theta.$$

- conversion to cylindrical coordinates:

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$$

- conversion to spherical coordinates:

$\left|\dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right| \rightarrow$ abs. value det of 3×3 matrix

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iiint_{E^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

We call the equations that define the change of variables a **transformation:**

$$x = x(u,v), \quad y = y(u,v).$$

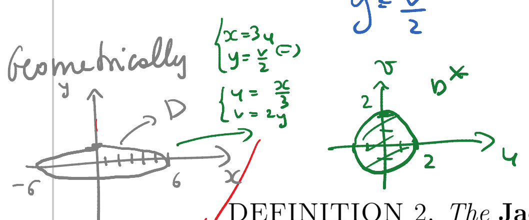
EXAMPLE 1. *Determine the new region that we get by applying the transformation $x = 3u, y = v/2$ to the region* $D = \underbrace{\left\{(x,y) |\ \frac{x^2}{36} + y^2 \leq 1\right\}}_{\text{the region enclosed by the ellipse } \frac{x^2}{36}+y^2=1}$.

Plug $\begin{matrix} x=3u \\ y=\frac{v}{2}\end{matrix}$ into $\frac{x^2}{36}+y^2\leq 1$: $\frac{(3u)^2}{36}+\left(\frac{v}{2}\right)^2=1 \iff$

$\underbrace{\frac{9}{36}}_{\frac{1}{4}}u^2+\frac{1}{4}v^2=1 \iff u^2+v^2\leq 4$

$D^*=\{(u,v):\ u^2+v^2\leq 4\} \rightarrow$ the disk of radius 2

Geometrically



compress 3 times along x-axis and stretch 2 times along y-axis

DEFINITION 2. *The* **Jacobian** *of the transformation $x = x(u,v), y = y(u,v)$ is*

$$\frac{\partial(x,y)}{\partial(u,v)} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}}_{\text{determinant}} \begin{matrix} \rightarrow \text{grad } x \\ \rightarrow \text{grad } y \end{matrix}$$

EXAMPLE 3. *Compute the Jacobian of the transformation $x = r\cos\theta,\ y = r\sin\theta$.*

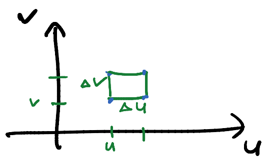
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

exactly the factor of change of area element in polar

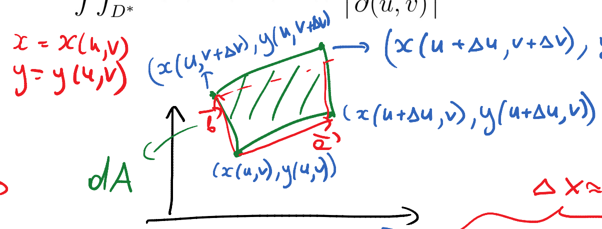
Change of variables for a double integral: → absolute value of Jacobian

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v.$$

Explanation



$$\xrightarrow[y = y(u,v)]{x = x(u,v)}$$



Recall the area of the parallelogram generated by vectors $\vec{a} = \langle a_1,a_2\rangle$ & $\vec{b} = \langle b_1, b_2\rangle$ is equal to $\left|\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2\end{pmatrix}\right|$ → consequence of the properties of cross product.

In our case $\vec{a} = \langle \underbrace{x(u+\Delta u,v)-x(u,v)}_{\Delta x \approx dx}, \underbrace{y(u+\Delta u,v)-y(u,v)}_{\Delta y\approx dy}\rangle \approx$

$\approx \langle \frac{\partial x}{\partial u}(u,v)\Delta u, \frac{\partial y}{\partial u}(u,v)\Delta u\rangle$ (+ terms of higher order in Δu)

using approxim. by the differential

$\vec{b} = \langle x(u,v+\Delta v)-x(u,v), y(u,v+\Delta v)-y(u,v)\rangle \approx$

$\approx \langle \frac{\partial x}{\partial v}(u,v)\Delta v, \frac{\partial y}{\partial v}(u,v)\Delta v\rangle$

$$dA \approx \left|\det\begin{pmatrix} \frac{\partial x}{\partial u}(u,v)\Delta u & \frac{\partial x}{\partial v}(u,v)\Delta v \\ \frac{\partial y}{\partial u}(u,v)\Delta u & \frac{\partial y}{\partial v}(u,v)\Delta v\end{pmatrix}\right| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \underbrace{\Delta u\Delta v}_{du\,dv}$$

EXAMPLE 4. *Evaluate*

$$\iint_D e^{\frac{y-x}{y+x}}\,\mathrm{d}A$$

where D is triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$.

$u = y-x$
$v = y+x$

We need to find $\frac{\partial(x,y)}{\partial(u,v)}$

This is convenient to do using the inverse Jacobian

EXAMPLE 5. *Find mass of a lamina that occupies the region*

$$D = \left\{(x, y) | 16 \leq x^2 + y^2 \leq 25, \quad 1 \leq x^2 - y^2 \leq 9, \quad x \geq 0, y \geq 0\right\}$$

with density $\rho(x, y) = 8xy$.