



F19_LN_1...

16.1: Vector Fields

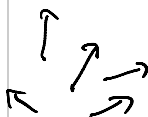
A vector function

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is an example of a function whose domain is a set of real numbers and whose range is a set of vectors in \mathbb{R}^3 :

$$\mathbf{r}(t) : \mathbb{R} \rightarrow \mathbb{R}^3.$$

A vector fields on \mathbb{R}^2 (or \mathbb{R}^3) is an assignment so that to every point of \mathbb{R}^2 (or of \mathbb{R}^3) one assigns a vector in \mathbb{R}^2 (or \mathbb{R}^3). Usually one attaches the starting point of this vector to the point to which this vector is assigned.



Vector field on \mathbb{R}^2 :

An assignment of a vector $\vec{F}(x,y)$ to every point (x,y)

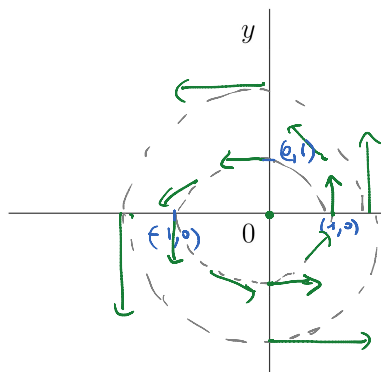
$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y) \rangle$$

Vector field on \mathbb{R}^3 :

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

A vector field in the plane (for instance), can be visualized as a collection of arrows with a given magnitude and direction, each attached to a point in the plane.

EXAMPLE 1. Describe the vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ by sketching.



$\vec{F}(x,y)$	$\vec{F}(x,y)$
$(0,0)$	$\langle 0,0 \rangle$ → stationary (or critical) point
$(1,0)$	$-0\hat{i} + 1\hat{j} = \langle 0,1 \rangle$
$(0,1)$	$\langle -1,0 \rangle$
$(-1,0)$	$\langle 0,-1 \rangle$
$(0,-1)$	$\langle 1,0 \rangle$
$(1,1)$	$\langle -1,1 \rangle$

not conservative because moving along the gradient increases the value and we cannot return to the same point.

From the point of view of diff. eq.

$\begin{cases} f_x = P \\ f_y = Q \\ f_z = R \end{cases} \rightarrow 3 \text{ eq. for one function - usually no solutions}$
 overdetermined system
 # eq > # unknown
 (There should be some compatibility relations between P, Q, R (namely $\text{curl } \vec{F} = 0$) in order that f exists)
 For instance, the vector field $\mathbf{F}(x, y) = \langle x, y \rangle$ is a conservative vector field with a potential function $f(x, y) = xy$ because

$$\nabla f = \langle f_x, f_y \rangle = \langle y, x \rangle$$

For $\mathbf{F}(x, y) = \langle x, y \rangle$ we can take $f = \frac{1}{2}(x^2 + y^2)$
 $\nabla f = \langle x, y \rangle$

REMARK 5. Not all vector fields are conservative, but such fields do arise frequently in Physics.

EXAMPLE 6. (see Example 2) Let

$$f(x, y, z) = \frac{GmM}{r},$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Find its gradient and answer the questions:

(a) Is the gravitational field conservative?

(b) What is a potential function of the gravitational field?

(a) $\nabla f?$ $f = \frac{GmM}{(x^2 + y^2 + z^2)^{1/2}} = GmM (x^2 + y^2 + z^2)^{-1/2}$

$$f_x = -\frac{1}{2} GmM (x^2 + y^2 + z^2)^{-3/2} \cdot 2x = -GmM \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = -GmM \frac{x}{r^3}$$

$$f_y = \dots = -GmM \frac{y}{r^3}, \quad f_z = \dots = -GmM \frac{z}{r^3}$$

$$\nabla f = -\frac{GmM}{r^3} \cdot \langle \underbrace{x, y, z}_{\vec{r}} \rangle = -\frac{GmM}{r^3} \vec{r} = \text{Gravitational vector field see example 2}$$

