

16.1: Vector Fields

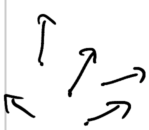
A vector function

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

is an example of a function whose domain is a set of real numbers and whose range is a set of vectors in $\mathbb{R}^3$:

$$\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3.$$

A vector fields on $\mathbb{R}^2$ (or $\mathbb{R}^3$) is an assignment so that to every point of $\mathbb{R}^2$ (or of $\mathbb{R}^3$) one assigns a vector in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Usually one attaches the starting point of this vector to the point to which this vector is assigned.



Vector field on $\mathbb{R}^2$: An assignment of a vector $\vec{F}(x,y)$ to every point (x,y)

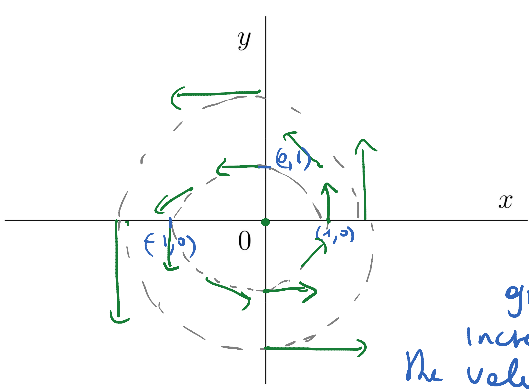
$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = \langle P(x,y), Q(x,y)\rangle$$

Vector field on $\mathbb{R}^3$:

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

A vector field in the plane (for instance), can be visualized as a collection of arrows with a given magnitude and direction, each attached to a point in the plane.

EXAMPLE 1. *Describe the vector field* $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$ *by sketching.*



not conser-vative, because moving along the gradient increases the value and we cannot return to the same point.

$\vec{r} = (x,y)$	$\vec{F}(x,y)$
$(0,0)$	$\langle 0,0\rangle$ → stationary (or critical) point
$(1,0)$	$-0\hat{i}+1\hat{j} = \langle 0,1\rangle$
$(0,1)$	$\langle -1,0\rangle$
$(-1,0)$	$\langle 0,-1\rangle$
$(0,-1)$	$\langle 1,0\rangle$
$(1,1)$	$\langle -1,1\rangle$

From the point of view of diff. eq.

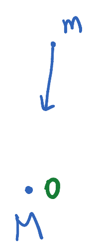
Same point.

From the point of view of diff. eq.
this vector field is tangent to circles around O
and represent motions with angular velocity 1 counter-clockwise.

Vector fields are often used to model, for example, the speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from one point to another point.

EXAMPLE 2. **Gravitational Field:**

By Newton's Law of Gravitation the magnitude of the gravitational force between two objects with masses m and M is ~~The gravitational force acting on the object at (x, y, z) is~~



$$|\mathbf{F}| = G\frac{mM}{r^2},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance between the objects and G is the gravitational constant.

$$\vec{F} = G\frac{mM}{r^2}\cdot\left(-\frac{\vec{r}}{\underbrace{|\vec{r}|}_{r}}\right) = -G\frac{mM}{r^3}\vec{r} = \left\langle -GmM\frac{x}{(x^2+y^2+z^2)^{3/2}}, -GmM\frac{y}{(x^2+y^2+z^2)^{3/2}}, -GmM\frac{z}{(x^2+y^2+z^2)^{3/2}}\right\rangle$$

$\vec{r} = \langle x, y, z\rangle$ position vector

Function $u = f(x, y, z)$ is also called a **scalar field**. Its gradient is also called **gradient vector field**:

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z) = \langle f_x, f_y, f_z\rangle$$

EXAMPLE 3. *Find the gradient vector field of $f(x, y, z) = xyz$.*

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z\rangle = \langle yz, xz, xy\rangle$$

In Physics a potential is a function f s.t $\vec{F} = -\nabla f$

DEFINITION 4. *A vector field* **F** *is called a* **conservative vector field** *if it is the gradient of some scalar function f s.t* $\mathbf{F} = \nabla f$. *In this situation f is called a* **potential function** *for* **F**.

Conservative vector fields are very rare.

$\vec{F} = \langle P, Q, R\rangle$ is conservative if there exists f s.t.

$$\begin{cases} f_x = P \\ f_y = Q \\ f_z = R \end{cases} \to \text{3 eq. for one function — usually no solutions}$$

(... conservative fields relations

$$\begin{cases} f_x = P \\ f_y = Q \\ f_z = R \end{cases} \to \text{3 eq. for one function — usually no solutions}$$

overdetermined system

\# eq > \# unknown

(There should some compatibility relations between P, Q, R (namely $\text{curl}\,\vec{F} = 0$) in order that f exists)

For instance, the vector field $\mathbf{F}(x, y) = \cancel{\langle x, y\rangle}$ $\langle y, x\rangle$ is a conservative vector field with a potential function $f(x, y) = xy$ because

$$\nabla f = \langle f_x, f_y\rangle = \langle y, x\rangle$$

For $F(x,y) = \langle x, y\rangle$ we can take $f = \frac{1}{2}(x^2+y^2)$

$$\nabla f = \langle x, y\rangle$$

REMARK 5. Not all vector fields are conservative, but such fields do arise frequently in Physics.

EXAMPLE 6. *(see Example 2)Let*

$$f(x, y, z) = \frac{GmM}{r},$$

where $r = \sqrt{x^2 + y^2 + z^2}$. *Find its gradient and and answer the questions:*

(a) *Is the gravitational field conservative?*

(b) *What is a potential function of the gravitational field?*

(a) ∇f?

$$f = \frac{GmM}{(x^2+y^2+z^2)^{1/2}} = GmM\,(x^2+y^2+z^2)^{-1/2}$$

$$f_x = -\frac{1}{\cancel{2}}GmM\,(x^2+y^2+z^2)^{-3/2}\cdot\cancel{2}x = -GmM\frac{x}{(x^2+y^2+z^2)^{3/2}} = -GmM\frac{x}{r^3}$$

$$f_y = \dots = -GmM\frac{y}{r^3}, \quad f_z = \dots = -GmM\frac{z}{r^3}$$

$$\nabla f = -\frac{GmM}{r^3}\cdot\underbrace{\langle x, y, z\rangle}_{\vec{r}} = -\frac{GmM}{r^3}\vec{r} = \text{Gravitational vector field}$$

see example 2