F19_LN_16_2

Sunday, November 10, 2019 11:56 PM

F19_LN_1...

16.2: Line Integrals

Line integrals on plane: Let C be a plane curve with parametric equations:

$$x = x(t), y = y(t), \quad a \leq t \leq b,$$

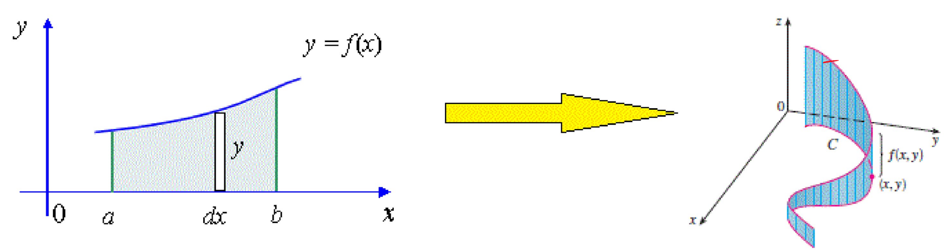
or we can write the parametrization of the curve as a vector function:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad a \leq t \leq b.$$

DEFINITION 1. *The line integral of $f(x, y)$ with respect to arc length, or the* **line integral of f along** C *is*

$$\int_C f(x, y)\, \mathrm{d}s$$

length element on C



Definite integral - Area of a flat surface

Line integral - Area of a curved surface

https://brilliant.org/wiki/line-integral/

Recall that the *arc length* of a curve given by parametric equations $x = x(t), y = y(t), \quad a \leq t \leq b$ can be found as

$$L = \int_a^b \mathrm{d}s,$$

where

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t.$$

The line integral is then

$$\int_C f(x, y)\, \mathrm{d}s = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

For length we took $f \equiv 1$

If we use the vector form of the parametrization we can simplify the notation up noticing that

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

and then

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t =$$

Using this notation the line integral becomes and then

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt =$$

Using this notation the line integral becomes,

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\, dt.$$

REMARK 2. The value of the line integral does not depend on the parametrization of the curve, provided that *the curve is traversed exactly once as t increases from a to b.*

Let us emphasize that $\quad ds = |r'(t)|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$

EXAMPLE 3. *Evaluate the line integral* $\int_C y\, ds$, *where* $C : x = t^3, y = t^2, 0 \le t \le 1.$

$$\int_C y\, ds = \int_0^1 t^2 \cdot \sqrt{\underbrace{(3t^2)^2}_{(x'(t))^2} + \underbrace{(2t)^2}_{(y'(t))^2}}\, dt = \int_0^1 t^2 \sqrt{9t^4 + 4t^2}\, dt =$$

$$= \int_0^1 t^2 \cdot t \sqrt{9t^2 + 4}\, dt = \frac{1}{18}\int_4^{13} \frac{u-4}{9}\sqrt{u}\, du = \frac{1}{18 \cdot 9}\int_4^{13} (u^{3/2} - 4u^{1/2})\, du = \overset{\text{Calc 1}}{\ldots}$$

$u = 9t^2 + 4 \xrightarrow{t: 0 \to 1} u: 4 \to 13$

$du = 18t\, dt \Rightarrow t\, dt = \frac{1}{18} du$

$\to t^2 = \frac{u-4}{9}$

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or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

The **line integral of f along C** is

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))|r'(t)|\, dt.$$

Here

$$ds = |r'(t)|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

EXAMPLE 4. *Evaluate the line integral* $\int_C (x + y + z)\, ds$, *where C is the line segment joining the points* $A(-1, 1, 2)$ *and* $B(2, 3, 1)$.

Parametrize the segment: $\overrightarrow{AB} = \langle 3, 2, -1 \rangle$

Parametric $\quad x = -1 + 3t$ $\quad \Rightarrow ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt$

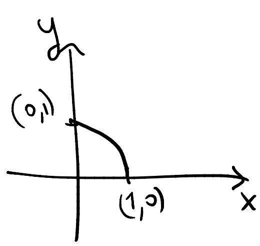
Parametric equation is:
$$\begin{cases} x = -1+3t \\ y = 1+2t, \quad 0 \le t \le 1 \\ z = 2-t \end{cases} \Rightarrow ds = \sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt = \sqrt{9+4+1}\,dt = \sqrt{14}\,dt$$

$$\int_C (x+y+z)\,ds = \int_0^1 \big((-1+3t)+(1+2t)+(2-t)\big)\sqrt{14}\,dt = \sqrt{14}\int_0^1 (2+4t)\,dt =$$
$$= \sqrt{14}\,(2+2) = \boxed{4\sqrt{14}}$$

Physical interpretation of a line integral: Let $\rho(x,y,z)$ represents the linear density at a point (x,y,z) of a thin wire shaped like a curve C. Then the **mass** m of the wire is:

$$m = \int_C \rho(x,y,z)\,\mathrm{d}s.$$

EXAMPLE 5. *A thin wire with the linear density $\rho(x,y) = x^2+2y^2$ takes the shape of the curve C which consists of the arc of the circle $x^2+y^2=1$ from $(1,0)$ to $(0,1)$* counterclockwise. *Find the mass of the wire.*



Parametrize the curve C

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \quad 0 \le \theta \le \frac{\pi}{2}$$

$$ds = \sqrt{x'(\theta)^2+y'(\theta)^2}\,d\theta = \sqrt{\underbrace{(-\sin\theta)^2+(\cos\theta)^2}_{1}}\,d\theta = d\theta$$

$$m = \int_S \rho(x,y)\,ds = \int_0^{\frac{\pi}{2}} \underbrace{\overbrace{(\cos^2\theta+2\sin^2\theta)}^{\cos^2\theta+\sin^2\theta+\sin^2\theta}}_{=\rho(\cos\theta,\sin\theta)}\,d\theta = \int_0^{\frac{\pi}{2}} (1+\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}})\,d\theta =$$

$$= \int_0^{\pi/2} \frac{3}{2}\,d\theta - \underbrace{\frac{1}{2}\int_0^{\pi/2} \cos 2\theta\,d\theta}_{\frac{1}{2}\sin 2\theta\,\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = 0} = \frac{3}{2}\cdot\frac{\pi}{2} = \boxed{\frac{3\pi}{4}}$$

Line integrals with respect to x, y, and z. Let C be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

The **line integral of f with respect to x** is,

$$\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$$

The **line integral of f with respect to** x is,

$$\int_C f(x,y,z)\,\mathrm{d}x = \int_a^b f\big(x(t),y(t),z(t)\big)x'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** y is,

$$\int_C f(x,y,z)\,\mathrm{d}y = \int_a^b f(x(t),y(t),z(t))y'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** z is,

$$\int_C f(x,y,z)\,\mathrm{d}z = \int_a^b f\big(x(t),y(t),z(t)\big)z'(t)\,dt$$

These integrals often appear together by the following notation:

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz$$

or

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y. = \int_C P\,dx + \int_C Q\,dy$$

$$\vec{F} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$$

EXAMPLE 6. *Compute*

$$I = \int_C -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y,$$

where C *is the circle* $x^2+y^2=1$ *oriented in the counterclockwise direction.*

Parametrize C: $\quad x = \cos\theta,\ y = \sin\theta, \quad 0 \le \theta \le 2\pi \qquad x'(\theta) = -\sin\theta,\ y'(\theta) = \cos\theta$

$$I = \int_0^{2\pi}\Bigg(\underbrace{-\frac{\sin\theta}{\cos^2\theta+\sin^2\theta}}_{-\frac{y(\theta)}{x^2(\theta)+y^2(\theta)}} \cdot \underbrace{(-\sin\theta)}_{x'(\theta)} + \frac{\cos\theta}{\underbrace{\cos^2\theta+\sin^2\theta}_{1}} \cdot \underbrace{\cos\theta}_{y'(\theta)}\Bigg)\,d\theta =$$

$$= \int_0^{2\pi}\big(\underbrace{\sin^2\theta + \cos^2\theta}_{1}\big)\,d\theta = \int_0^{2\pi} d\theta = 2\pi$$

Line integrals of vector fields.

PROBLEM: Given a continuous force field,

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k},$$

such as a gravitational field. Find the work done by the force $\mathbf{F}$ in moving a particle along a curve

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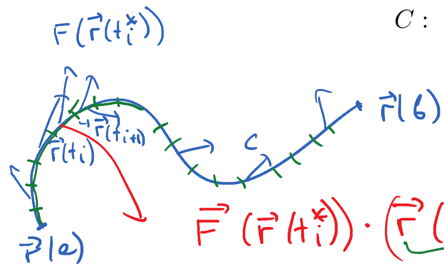
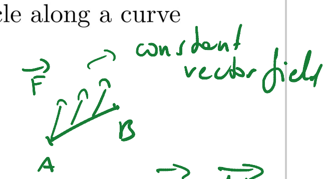
$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b.$$





$W = \vec{F} \cdot \vec{AB}$

$\vec{F}(\vec{r}(t_i^*)) \cdot (\underbrace{\vec{r}(t_{i+1}) - \vec{r}(t_i)}$

$= \vec{r}\,'(t_i)\underbrace{(t_{i+1} - t_i)}_{\Delta t_i} + O(\Delta t_i^2)$

$$\lim_{\Delta t_i \to 0} \sum \vec{F}(\vec{r}(t_i^*))(\vec{r}(t_{i+1}) - \vec{r}(t_i)) = \int_a^b \overbrace{\vec{F}(\vec{r}(t))}^{\langle P, Q, R\rangle} \cdot \overbrace{\vec{r}\,'(t)}^{\langle x', y', z'\rangle}\, dt =$$

$$\| \quad \lim_{\Delta t_i \to 0} \sum \vec{F}(\vec{r}(t_i^*)) \cdot r'(t_i)\Delta t_i =$$

$$= \int_a^b P(x(t), y(t), z(t))\, x'(t) + Q(x(t), y(t), z(t)) \cdot y'(t) + R(x(t), y(t), z(t))\, z'(t)\, dt$$

DEFINITION 7. *Let* $\mathbf{F}$ *be a continuous vector field defined on a curve* C *given by a vector function* $\mathbf{r}(t)$, $a \le t \le b$. *Then the* **line integral of F along** C *is* $= \int_C P\,dx + Q\,dy + R\,dz$

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t.$$

REMARK 8. Note that this integral depends on the curve orientation:

$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t)$$

Relationship between line integrals of vector fields and line integrals with respect to x, y, **and** z.

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = \int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = \int_C P\,dx + Q\,dy + R\,dz$$

EXAMPLE 9. *Find the work done by the force field* $\mathbf{F}(x,y,z) = \langle \underbrace{xy}_{P}, \underbrace{yz}_{Q}, \underbrace{xz}_{R}\rangle$ *in moving a particle along the curve* $C : \mathbf{r}(t) = \langle t, t^2, t^3\rangle, 0 \le t \le 1$.

$x = t,\ y = t^2,\ z = t^3$

$x'(t) = 1,\ y'(t) = 2t,\ z'(t) = 3t^2$

$$W = \int^1 \langle t \cdot t^2, t^2 \cdot t^3, t \cdot t^3\rangle \cdot \langle 1, 2t, 3t^2\rangle\, dt =$$

$$x'(t)=1 \,,\; y'(t)=2t \,,\; z'(t)=3t$$

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$$\int_0^1 \langle P, Q, R \rangle \cdot \langle x', y', z' \rangle \, dt$$

$$= \int_0^1 \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt = \int_0^1 \left(t^3 + 2t^6 + 3t^6 \right) dt =$$

$$= \int_0^1 \left(t^3 + 5t^6 \right) dt = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$

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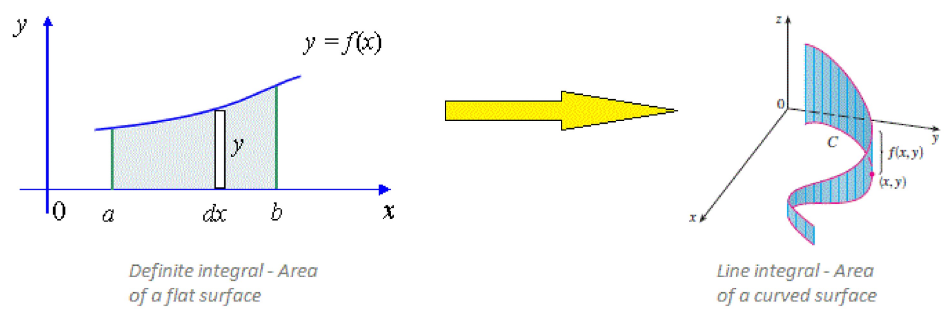

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Using this notation the line integral becomes,

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