# F19_LN_16_2

Sunday, November 10, 2019 11:57 PM

F19_LN_1...



## 16.2: Line Integrals

**Line integrals on plane:** Let $C$ be a plane curve with parametric equations:

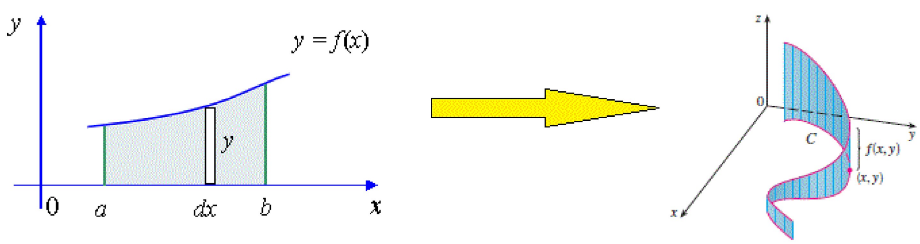
$$x = x(t), y = y(t), \quad a \le t \le b,$$

or we can write the parametrization of the curve as a vector function:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad a \le t \le b.$$

DEFINITION 1. *The line integral of $f(x, y)$ with respect to arc length, or the* **line integral of $f$ along** *$C$ is*

$$\int_C f(x,y)\,\underbrace{\mathrm{d}s}_{\text{length element on } C}$$




*Definite integral - Area of a flat surface*

*Line integral - Area of a curved surface*

*https://brilliant.org/wiki/line-integral/*

Recall that the *arc length* of a curve given by parametric equations $x = x(t), y = y(t), \quad a \le t \le b$ can be found as

$$L = \int_a^b \mathrm{d}s,$$

where

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

For the length we took $f \equiv 1$

The line integral is then

$$\int_C f(x,y)\,\mathrm{d}s = \int_a^b f\big(x(t), y(t)\big)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

If we use the vector form of the parametrization we can simplify the notation up noticing that

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$$

and then

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t = |\mathbf{r}'(t)|\,dt$$

Using this notation the line integral becomes,

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = |r'(t)|\, dt$$

Using this notation the line integral becomes,

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\, dt.$$

REMARK 2. The value of the line integral does not depend on the parametrization of the curve, provided that *the curve is traversed exactly once as $t$ increases from $a$ to $b$.* → consequence of a chain rule.



Let us emphasize that $\quad ds = |r'(t)|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$

EXAMPLE 3. *Evaluate the line integral* $\int_C y\, ds$, *where* $C : x = t^3, y = t^2, 0 \le t \le 1.$

$x' = 3t^2, \; y' = 2t$

$$\int_C y\, ds = \int_0^1 \underbrace{t^2}_{y} \sqrt{\underbrace{9t^4}_{(x'(t))^2} + \underbrace{4t^2}_{(y'(t))^2}}\, dt = \int_0^1 \underbrace{t^2}_{\frac{u-4}{9}} \cdot t \underbrace{\sqrt{9t^2+4}}_{\sqrt{u}}\, dt =$$

$u = 9t^2 + 4$ $\quad$ $t: 0 \to 1$

$du = 18t\, dt$ $\quad$ $u: 4 \to 13$

$t\, dt = \frac{1}{18} du$

$$= \frac{1}{18} \int_4^{13} \frac{u-4}{9} \cdot \sqrt{u}\, du = \frac{1}{162} \int_4^{13} (u^{3/2} - 4u^{1/2})\, du = \ldots \text{ Calc 1}$$

**Line integrals in space:** Let $C$ be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a \le t \le b.$$

The **line integral of $f$ along $C$** is

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))|r'(t)|\, dt.$$

Here

$$ds = |r'(t)|\, dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

EXAMPLE 4. *Evaluate the line integral* $\int_C (x+y+z)\, ds$, *where $C$ is the line segment joining the points* $A(-1,1,2)$ *and* $B(2,3,1)$.

Parametrize $C$: $\vec{AB} = \langle 3, 2, -1 \rangle$

$C$ can be parametrized by: $\begin{cases} x = -1 + 3t \\ y = 1 + 2t \\ z = 2 - t \end{cases}$ $\quad 0 \le t \le 1$

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$$r'(t) = \langle 3, 2, -1 \rangle \Rightarrow ds = |r'(t)|\,dt = \sqrt{9+4+1}\,dt = \sqrt{14}\,dt$$

$$\int_C (x+y+z)\,ds = \int_0^1 \Big(\underbrace{(-1+3t)}_{x} + \underbrace{(1+2t)}_{y} + \underbrace{(2-t)}_{z}\Big)\sqrt{14}\,dt = \sqrt{14}\int_0^1 (2+4t)\,dt =$$

$$= \sqrt{14}\left(2 + 4\cdot\frac{1}{2}\right) = \boxed{4\sqrt{14}}$$



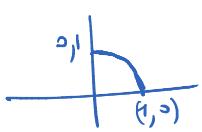
**Physical interpretation of a line integral**: Let $\rho(x, y, z)$ represents the linear density at a point $(x, y, z)$ of a thin wire shaped like a curve $C$. Then the **mass** $m$ of the wire is:

$$m = \int_C \rho(x, y, z)\,\mathrm{d}s.$$

EXAMPLE 5. *A thin wire with the linear density $\rho(x, y) = x^2 + 2y^2$ takes the shape of the curve $C$ which consists of the arc of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$* counterclockwise*. Find the mass of the wire.*

Solution Parametrize this arc by the angle:

$$x = \cos\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$
$$y = \sin\theta$$




$$ds = \sqrt{(x')^2 + (y')^2}\,d\theta = \underbrace{\sqrt{(-\sin\theta)^2 + (\cos\theta)^2}}_{1}\,d\theta = d\theta$$

$$m = \int_0^{\frac{\pi}{2}} \Big(\overbrace{\underbrace{\cos^2\theta + 2\sin^2\theta}_{1+\sin^2\theta}}^{\cos^2\theta + \sin^2\theta + \sin^2\theta = 1}\Big)\,d\theta = \int_0^{\frac{\pi}{2}} \Big(1 + \underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\Big)\,d\theta = \int_0^{\frac{\pi}{2}} \frac{3}{2}\,d\theta - \frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}} \cos 2\theta\,d\theta}_{\frac{1}{4}\sin 2\theta\,\big|_0^{\frac{\pi}{2}} = 0}$$

$$= \frac{3}{2}\cdot\frac{\pi}{2} = \boxed{\frac{3\pi}{4}}$$

**Line integrals with respect to $x, y$, and $z$.** Let $C$ be a space curve with parametric equations:

$$x = x(t), y = y(t), z = z(t), \quad a \le t \le b,$$

The **line integral of f with respect to $x$** is,

$$\int_C f(x, y, z)\,\mathrm{d}x = \int_a^b f(x(t), y(t), z(t))x'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to $y$** is,

$$\int_C f(x,y,z)\,\mathrm{d}x = \int_a^b f(x(t),y(t),z(t))x'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** $y$ is,

$$\int_C f(x,y,z)\,\mathrm{d}y = \int_a^b f(x(t),y(t),z(t))y'(t)\,\mathrm{d}t.$$

The **line integral of f with respect to** $z$ is,

$$\int_C f(x,y,z)\,\mathrm{d}z = \int_a^b f(x(t),y(t),z(t))\,z'(t)\,dt$$

These integrals often appear together by the following notation:

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz$$

or

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y. = \int_C P\,dx + \int_C Q\,dy$$

Rem



$$\operatorname{grad}(\underbrace{\arctan \tfrac{y}{x}}_{\theta(x,y)}) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

EXAMPLE 6. *Compute*

$$I = \int_C -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y,$$

*where $C$ is the circle $x^2+y^2=1$ oriented in the counterclockwise direction.*

Parametrize the circle $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$ $0 \le \theta \le 2\pi$ $\quad x'(\theta) = -\sin\theta$, $y'(\theta) = \cos\theta$

$$I = \int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int_0^{2\pi}\Big[-\frac{\sin\theta}{\underbrace{\cos^2\theta+\sin^2\theta}_{1}}\cdot\underbrace{(-\sin\theta)}_{x'(\theta)} + \frac{\cos\theta}{\underbrace{\cos^2\theta+\sin^2\theta}_{1}}\cdot\underbrace{\cos\theta}_{y'(\theta)}\Big]d\theta = \int_0^{2\pi}(\underbrace{\sin^2\theta+\cos^2\theta}_{1})\,d\theta = \int_0^{2\pi} d\theta = \boxed{2\pi}$$

**Line integrals of vector fields.**

*PROBLEM:* Given a continuous force field,

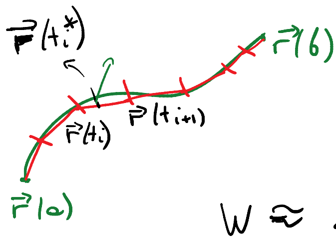
$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k},$$

such as a gravitational field. Find the work done by the force $\mathbf{F}$ in moving a particle along a curve

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \qquad a \le t \le b.$$

We want to calculate the work done by $\vec{F}$ in moving a particle along the curve $C$.

$\vec{F}(t_i^*)$ $\quad$ $\vec{r}(b)$




We want to calculate the work done by $\vec{F}$ in moving a particle along ~~the~~ the curve $C$.

Recall $\int\int \to$ constant vector field

$W = \vec{F} \cdot \vec{AB}$

$$W \approx \sum_i \vec{F}(\vec{r}(t_i^*)) \cdot (\vec{r}(t_{i+1}) - \vec{r}(t_i)) \approx$$

$$\vec{r}\,'(t_i)\underbrace{(t_{i+1}-t_i)}_{\Delta t_i} + o(\Delta t_i)$$

the ratio of this with $\Delta t_i$ goes to 0 as $\Delta t_i \to 0$

$$\approx \sum_i \vec{F}(\vec{r}(t_i^*))\,\vec{r}\,'(t_i)\Delta t_i \xrightarrow[\Delta t_i \to 0]{} \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt =$$

DEFINITION 7. *Let* $\mathbf{F}$ *be a continuous vector field defined on a curve* $C$ *given by a vector function* $\mathbf{r}(t)$, $a \le t \le b$. *Then the* **line integral of F along** $C$ *is*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t.$$

REMARK 8. Note that this integral depends on the curve orientation:

$$\int_{-C} \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = -\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t)$$



**Relationship between line integrals of vector fields and line integrals with respect to** $x, y,$ **and** $z$**.**

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}(t) = \int \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = \int_C P\,dx + Q\,dy + R\,dz$$

$$= \int_a^b \langle P, Q, R\rangle \cdot \langle x', y', z'\rangle\,dt$$

$$\underset{\text{after parametrization}}{=} \int_a^b \Big[P(x(t), y(t), z(t))\,x'(t) + Q(x(t), y(t), z(t))\,y'(t) + R(x(t), y(t), z(t))\,z'(t)\Big]dt$$

EXAMPLE 9. *Find the work done by the force field* $\mathbf{F}(x, y, z) = \langle \underbrace{xy}_{P}, \underbrace{yz}_{Q}, \underbrace{xz}_{R}\rangle$ *in moving a particle along the curve* $C : \mathbf{r}(t) = \langle t, t^2, t^3\rangle, 0 \le t \le 1.$

$$W = \int_C \vec{F} \cdot d\vec{r}(t) = \int_0^1 \langle \underbrace{t \cdot t^2}_{t^3}, \underbrace{t^2 \cdot t^3}_{t^5}, \underbrace{t \cdot t^3}_{t^4}\rangle \underset{\text{dot product}}{\cdot} \langle 1, 2t, 3t^2\rangle\,dt =$$

$x(t) = t$, $y(t) = t^2$, $z(t) = t^3$ $\Rightarrow$ $x'(t) = 1$, $y'(t) = 2t$, $z'(t) = 3t^2$

$y(t) = t^2 \Rightarrow y'(t) = 2t$
$z(t) = t^3 \qquad z'(t) = 3t^2$

$$= \int_0^1 \left(t^3 + 2t^6 + 3t^6\right) dt = \int_0^1 (t^3 + 5t^6)\, dt = \frac{1}{4} + \frac{5}{7} = \boxed{\frac{27}{28}}$$