F19_LN_16_3_4

Sunday, November 17, 2019 5:14 PM

F19_LN_1...

16.3: The fundamental Theorem for Line Integrals

16.4: Green's Theorem

→ (from Sec. 16.1)

DEFINITION 1. *A vector field* **F** *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* **F**.

Recall Part 2 of the Fundamental Theorem of Calculus:

$$\int_a^b \underbrace{F'(x)}_{\frac{d}{dx}F(x)}\, \mathrm{d}x = F(b) - F(a),$$

where F' is continuous on $[a, b]$.

- **The fundamental Theorem for Line Integrals:** Let C be a smooth curve given by $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

$F(t) := f(\vec{r}(t))$

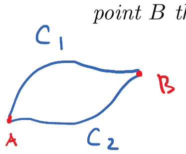
REMARK 2. If C is a closed curve then $\vec{r}(b) = \vec{r}(a)$

$$\oint_C \nabla f \cdot d\vec{r} = 0 \quad \left(\text{because } \vec{r}(b) = \vec{r}(a) \Rightarrow f(\vec{r}(b)) - f(\vec{r}(a)) = 0\right)$$

Proof: $f(\vec{r}(b)) - f(\vec{r}(a)) \underset{\text{Fund. Thm of Calc}}{=} \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt$

$$\underset{\text{chain rule}}{=} \int_a^b \underbrace{\frac{\partial f}{\partial x}(\vec{r}(t))\, x'(t) + \frac{\partial f}{\partial y}(\vec{r}(t))\, y'(t) + \frac{\partial f}{\partial z}(\vec{r}(t))\, z'(t)}_{\nabla f(\vec{r}(t)) \cdot \vec{r}'(t)}\, dt = \int_C \nabla f \cdot dr \quad \boxtimes$$

COROLLARY 3. *If* F *is a conservative vector field and* C *is a curve with initial point* A *and terminal point* B *then:*



$\int_C \vec{F} \cdot d\vec{r}$ is independent of a path C, but dependents on the initial and terminal point only:

If C_1 and C_2 are 2 paths with the same initial point A and terminal point B, then $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} = f(B) - f(A)$, where f is the potential of $\vec{F}$, i.e. $\nabla f = \vec{F}$

EXAMPLE 4. *Find the work done by the gravitational field*

$$\mathbf{F}(x, y, z) = -\frac{GmM}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle$$

in moving a particle with mass m *from the point* $(1, 2, 2)$ *to the point* $(3, 4, 12)$ *along a piecewise-smooth curve* C.

Based on Example 6 of sec. 16.1: $f = \frac{GmM}{(x^2+y^2+z^2)^{1/2}}$ is a potential for $\vec{F}$ $\Rightarrow$ $\vec{F}$ is conservative $\Rightarrow$

$$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = f(3,4,12) - f(1,2,2) =$$

$$= GmM\left(\frac{1}{\sqrt{3^2+4^2+12^2}} - \frac{1}{\sqrt{1+2^2+2^2}}\right) = \ldots$$

$$= GmM\left(\frac{1}{13} - \frac{1}{3}\right)$$

C is a path connecting $(1,2,2)$ with $(3,4,12)$

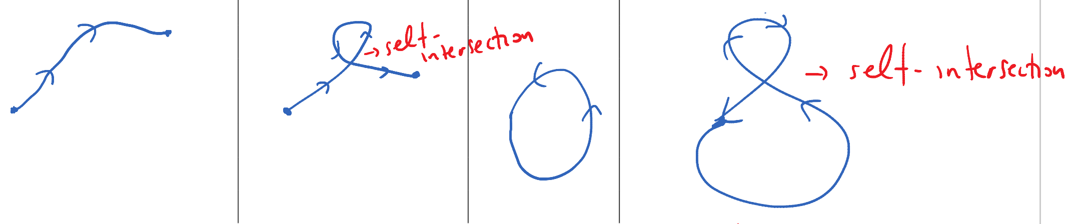
Notations And Definitions:

DEFINITION 5. *A piecewise-smooth curve is called a* **path**.

- **Types of curves:** self-intersection (inside)

  → without

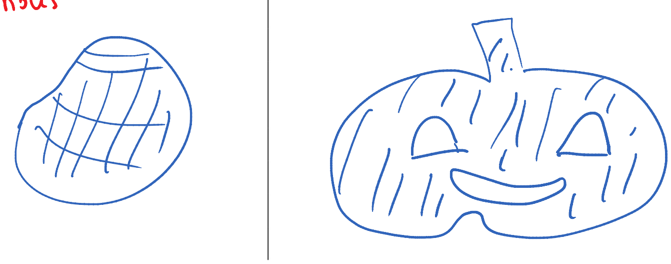
simple not closed	not simple not closed	simple closed	not simple, closed



- **Types of regions:** → there is at least one hole

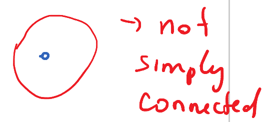
roughly no holes

simply connected	not simply connected

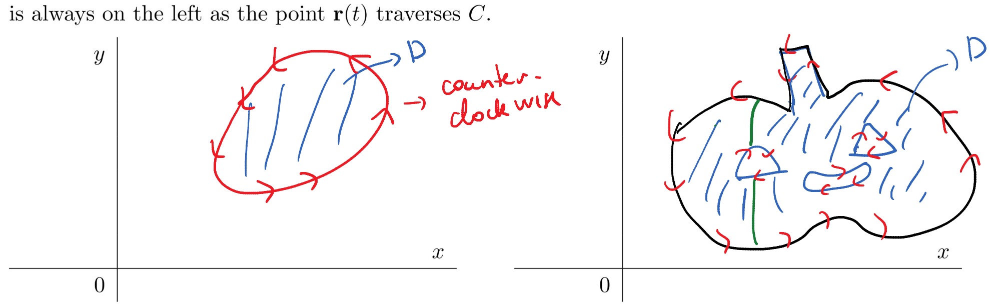


Another example

$\mathbb{R}^2 \setminus$ a point



- **Convention:** The **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}, a \le t \le b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C.



- The positively oriented boundary curve of D is denoted by ∂D.

•**GREEN's THEOREM**: *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then*

$$\left|\begin{matrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{matrix}\right| = \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P$$

the path is closed

$$\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A.$$

The rule is that we differentiate the y-component of the field w.r.t x and the x-component w.r.t y and the latter is multiply by −1.

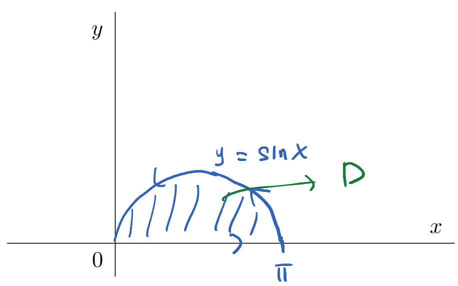
Rem The Green Thm is true for ~~not~~ simply connected D if ∂D is positively oriented

EXAMPLE 6. *Evaluate:*

positively oriented

$$I = \oint_C \underbrace{e^x(1-\cos y)}_{P}\,\mathrm{d}x \underbrace{- e^x(1-\sin y)}_{Q}\,\mathrm{d}y$$

where C is the boundary of the domain $D = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$.



Way 1 : Parametrize ☹

Way 2 Use Green's Thm ☺

$P = e^x(1-\cos y)$

$Q = -e^x(1-\sin y)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^x(1-\sin y) - e^x \sin y = -e^x + \cancel{e^x \sin y} - \cancel{e^x \sin y}$$

$$= -e^x \Rightarrow \underset{\text{Green Thm}}{I} = \iint_D (-e^x)\,dx\,dy = -\int_0^{\pi}\int_0^{\sin x} e^x\,dy\,dx = -\int_0^{\pi} e^x \cdot \sin x\,dx =$$

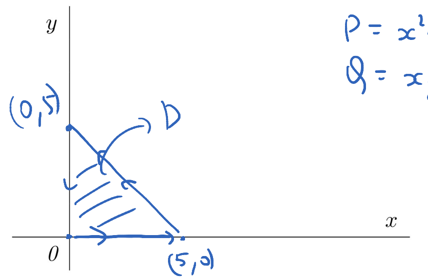
Calc 1 → integration by part using loops

$$= -\frac{1}{2}\left(e^{\pi}+1\right)$$

EXAMPLE 7. *Let C be a triangular curve consisting of the line segments from $(0,0)$ to (5,0), from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integral:*

$$I_1 = \oint_C \underbrace{(x^2y + \frac{1}{2}y^2)}_{P}\,\mathrm{d}x + \underbrace{(xy + \frac{1}{3}x^3 + 3x)}_{Q}\,\mathrm{d}y$$

Use Green's Thm



$P = x^2y + \frac{1}{2}y^2$

$Q = xy + \frac{1}{3}x^3 + 3x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \underbrace{\cancel{y} + \cancel{x^2} + 3}_{\frac{\partial Q}{\partial x}} - (\cancel{x^2} + \cancel{y}) = 3$$

$$I_1 \underset{\text{Green's Thm}}{=} \iint_D 3\,dx\,dy = 3\,\text{Area}(D) = 3 \cdot \frac{5 \cdot 5}{2} = \boxed{\frac{75}{2}}$$

SUMMARY: Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ *be a vector field on an open simply connected (no holes) domain D. Suppose that P and Q have continuous partial derivatives through D. Then the facts below are equivalent.*

True for arbitrary D (no need for simply connectedness)

The field **F** is **conservative**on D $\Longleftrightarrow$ There exists f s.t. $\nabla f = \mathbf{F}$

$\Rightarrow$ proved

$\Leftarrow$

The field **F** is **conservative** on D $\Longleftrightarrow$ $\int_{\overset{\smile}{AB}} \mathbf{F}\cdot \mathbf{dr}$ is independent of path in D

Fix a point A

Set $f(B) := \int_{AB} \vec{F}\cdot d\vec{r}$

Claim: $\nabla f = \vec{F}$

There exist f s.t.

$\begin{cases} \frac{\partial f}{\partial x} = P \xRightarrow{\text{Diff. w.r.t } y} f_{xy} = P_y \\ \frac{\partial f}{\partial y} = Q \xRightarrow{\text{Diff. w.r.t } x} f_{yx} = Q_x \end{cases} \Rightarrow P_y = Q_x$

The field **F** is **conservative** on D $\overset{\Rightarrow}{\underset{\Leftarrow}{\Longleftrightarrow}}$ $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ throughout D $\rightarrow$ at every point of D

$\Leftarrow$ if D is simply connected

True for arbitrary D

The field **F** is **conservative** on D $\Longleftrightarrow$ $\oint_C \mathbf{F}\cdot\mathbf{dr} = 0$ for every closed curve C in D

If D is not simply connected, then $Q_x = P_y$ in general does not imply that F is conservative (see Example 6 of 16.2)

EXAMPLE 8. *Determine whether or not the vector field is conservative:*

$D = \mathbb{R}^2 \to$ We check whether $Q_x = P_y$

(a) $\mathbf{F}(x,y) = \langle x^2+y^2, 2xy\rangle$.

$P = x^2+y^2$, $Q = 2xy$ $\quad P_y = 2y$, $Q_x = 2y$ $\Rightarrow Q_x = P_y \Rightarrow \vec{F}$ is conservative

(b) $\mathbf{F}(x,y) = \langle \underbrace{x^2+3y^2+2}_{P}, \underbrace{3x+ye^y}_{Q}\rangle$

$P = x^2+3y^2+2$ $\quad P_y = 6y$

$Q = 3x+ye^y$ $\quad Q_x = 3$

$\not\equiv \to$ not identically

$6y \not\equiv 3 \Rightarrow \vec{F}$ is not conservative

EXAMPLE 9. *Given* $\mathbf{F}(x,y) = \underbrace{\sin y}_{P}\mathbf{i} + \underbrace{(x\cos y + \sin y)}_{Q}\mathbf{j}$.

$D = \mathbb{R}^2 \to$ simply connected

(a) *Show that* $\mathbf{F}$ *is conservative.*

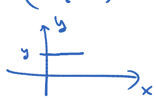
$P = \sin y$ $\quad P_y = \cos y$

$Q = x\cos y + \sin y$ $\quad Q_x = \cos y$

$P_y = Q_x \Rightarrow \vec{F}$ is conservative

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$ $\iff \begin{cases} f_x = P \\ f_y = Q \end{cases} \iff \begin{cases} f_x = \sin y & \text{(Eq 1)} \\ f_y = x\cos y + \sin y & \text{(Eq 2)} \end{cases}$

$$\text{(Eq 1)} \xRightarrow{\text{Integrate w.r.t. } x} f = \int \sin y\, dx = x\sin y + C(y) \quad (*)$$



Plug f into (Eq 2):

$$\frac{\partial}{\partial y}\left(x\sin y + C(y)\right) = \cancel{x\cos y} + C'(y) = \cancel{x\cos y} + \sin y \Rightarrow$$

Rem At this step we must have cancellation of all terms depending on x.

$$C'(y) = \sin y \Rightarrow C(y) = \int \sin y\, dy = -\cos y + C_1 \quad \xRightarrow{(*)}$$

$$f(x,y) = x\sin y - \cos y + C_1.$$ For example we can take $C_1 = 0$ $\Rightarrow f(x,y) = x\sin y - \cos y$

Rem You can start with Eq 2, Integrate w.r.t. y, get a function depending on $C(x)$ and plug into 1 eq.

(c) *Find the work done by the force field* $\mathbf{F}$ *in moving a particle from the point* $(3,0)$ *to the point* $(0,\pi/2)$.

$$\int_{C:(3,0)\to(0,\frac{\pi}{2})} \vec{F}\cdot d\vec{r} = f\left(0,\frac{\pi}{2}\right) - f(3,0) = \underbrace{\overset{=0}{0}\cdot\sin\frac{\pi}{2} - \overset{=0}{\cos\frac{\pi}{2}}}_{f(0,\frac{\pi}{2})} - \underbrace{\left(3\cdot\overset{=0}{\sin 0} - \overset{=1}{\cos 0}\right)}_{f(3,0)}$$

$$= 1$$

(d) *Evaluate* $\oint_C \mathbf{F}\,d\mathbf{r}$ *where* C *is an arbitrary* closed *path in* $\mathbb{R}^2$.

$$\oint_C \vec{F}\,dr = 0$$

→ will be done after 16.5

EXAMPLE 10. *Given*

$$\mathbf{F} = \langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \rangle .$$

Find a function f s.t. $\nabla f = \mathbf{F}$

1) Check if $\vec{F}$ is conservative by checking $\operatorname{curl} \vec{F} = 0$?

(if $\operatorname{curl} \vec{F} \neq 0$ then such f does not exist)

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3+z^2 & 3x^2y^2+2yz & y^2+2xz \end{vmatrix} = (\cancel{2y} - \cancel{2y})\,\hat{i} - (\cancel{2z} - \cancel{2z})\,\hat{j} + (\cancel{6xy^2} - \cancel{6xy^2})\,\hat{k} = 0$$

Since $\vec{F}$ is defined on $\mathbb{R}^3$ then $\operatorname{curl} \vec{F} = 0 \Rightarrow \vec{F}$ is conservative

let us find f: $\nabla f = \vec{F}$

$$\begin{cases} f_x = 2xy^3 + z^2 & (\text{Eq } 1) \\ f_y = 3x^2y^2 + 2yz & (\text{Eq } 2) \\ f_z = y^2 + 2xz & (\text{Eq } 3) \end{cases}$$

$\overset{\text{integrate w.r.t. } x}{\Longrightarrow}$ $f = \int (2xy^3 + z^2)\, dx = x^2y^3 + z^2x + C(y,z)$ (*)

Plug (*) into (Eq 2): $\frac{\partial}{\partial y}\left(x^2y^3 + z^2x + C(y,z)\right) = 3\cancel{x^2y^2} + \frac{\partial}{\partial y} C(y,z) = \cancel{3x^2y^2} + 2yz$

All terms with x must be cancelled at this step

$$\frac{\partial}{\partial y} C(y,z) = 2yz \overset{\text{Integrate w.r.t. } y}{\Longrightarrow} C(y,z) = \int 2yz\, dy = y^2z + C_1(z) \overset{\text{Plug to } (*)}{\Longrightarrow}$$

$$f = x^2y^3 + z^2x + y^2z + C_1(z) \overset{\text{Plug this}}{\underset{\text{to (Eq 3)}}{\Longrightarrow}}$$

$$\frac{\partial}{\partial z}\left(x^2y^3 + z^2x + y^2z + C_1(z)\right) = \cancel{2zx} + \cancel{y^2} + \frac{d}{dz} C_1(z) = \cancel{y^2} + \cancel{2xz}$$

$$\frac{d}{dz} C_1(z) = 0 \qquad C_1 \equiv \text{const} . \Rightarrow$$

$$\boxed{f = x^2y^3 + z^2x + y^2z} + C_1 .$$

We can take $C_1 = 0$