F19_LN_16_3_4

Sunday, November 17, 2019 5:15 PM

F19_LN_1...

16.3: The fundamental Theorem for Line Integrals

16.4: Green's Theorem

(discussed in Sec. 16.1)

DEFINITION 1. *A vector field* **F** *is called a* **conservative vector field** *if it is the gradient of some scalar function* f *s.t* $\mathbf{F} = \nabla f$. *In this situation* f *is called a* **potential function** *for* **F**.

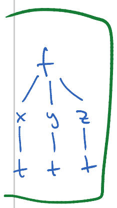
Recall Part 2 of the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a),$$

where F' is continuous on $[a, b]$.

• **The fundamental Theorem for Line Integrals:** Let C be a smooth curve given by $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables and ∇f is continuous on C. Then

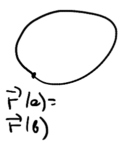
$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \quad (*) \qquad F(t) := f(\vec r(t))$$



REMARK 2. If C is a closed curve then

$\vec r(b) = \vec r(a)$

$\int_C \nabla f \cdot dr = \underbrace{0}_{f(\vec r(b)) - f(\vec r(a)) = 0}$



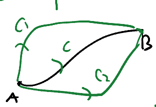
Proof of (*): $\underbrace{f(\vec r(b)) - f(\vec r(a))}_{F(b) - F(a)} \underset{\text{Fund. Thm of Calc}}{=}$

$= \int_a^b \frac{d}{dt} \underbrace{f(\vec r(t))}_{F(t)}\, dt \underset{\text{chain rule}}{=} \int_a^b \left(\frac{\partial f}{\partial x}(\vec r(t))\, x'(t) + \frac{\partial f}{\partial y}(\vec r(t))\, y'(t) + \frac{\partial f}{\partial z}(\vec r(t))\, z'(t)\right) dt = \int_a^b \nabla f(\vec r(t)) \cdot \vec r\,'(t)\, dt$ ∎

COROLLARY 3. *If* F *is a conservative vector field and* C *is a curve with initial point* A *and terminal point* B *then:*

$F = \nabla f$ for some f

$\int_C \vec F \cdot d\vec r(t) = \underbrace{f(B) - f(A)}_{\text{the difference of the values of a potential at } B \text{ and at } A} \longrightarrow$ Independent of a path C (depends on the initial and terminal point only)



$\int_{C_1} \vec F \cdot d\vec r = \int_{C_2} \vec F \cdot d\vec r = \int_C \vec F \cdot d\vec r$

EXAMPLE 4. *Find the work done by the gravitational field*

$$\mathbf{F}(x, y, z) = -\frac{GmM}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle$$

in moving a particle with mass m *from the point* $(1, 2, 2)$ *to the point* $(3, 4, 12)$ *along a piecewise-smooth curve* C.

By Example 6 of Sec 16.1 $\quad f(x,y,z) = \frac{GmM}{(x^2+y^2+z^2)^{1/2}}$ is a potential of $\vec F$, i.e. $\nabla f = \vec F \Rightarrow$

$W = \int_C \vec F \cdot d\vec r \overset{= \int_C \nabla f \cdot d\vec r =}{=} f(3,4,12) - f(1,2,2) = GmM\left(\frac{1}{\sqrt{3^2+4^2+12^2}} - \frac{1}{\sqrt{1^2+2^2+2^2}}\right)$

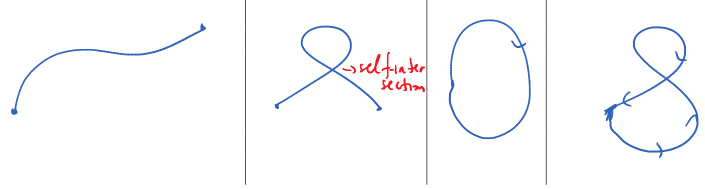
$= GmM\left(\frac{1}{13} - \frac{1}{3}\right)$

Notations And Definitions:

DEFINITION 5. *A piecewise-smooth curve is called a* **path**.

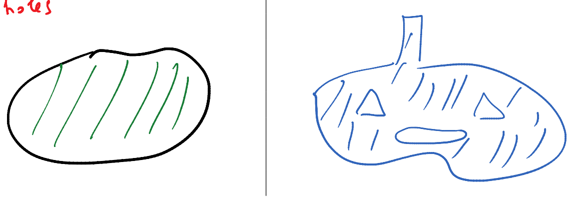
- **Types of curves:** without self - intersection

simple not closed	not simple not closed	simple closed	not simple, closed

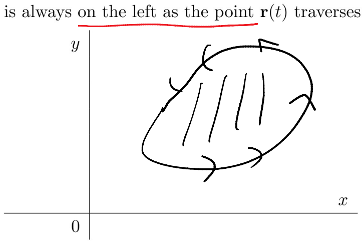


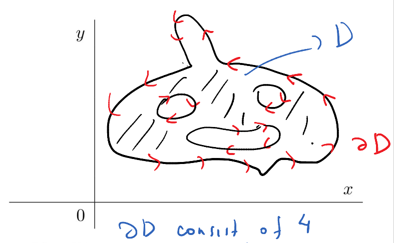
- **Types of regions:**

roughly no holes ← simply connected | not simply connected → roughly there is a hole (at least one)



- **Convention:** The **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. If C is given by $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}, a \le t \le b$, then the region D bounded by C is always on the left as the point $\mathbf{r}(t)$ traverses C.

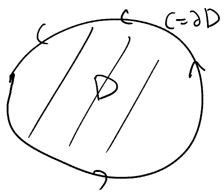




- The positively oriented boundary curve of D is denoted by ∂D.

∂D consist of 4 simple closed curves

•**GREEN's THEOREM**: *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region that contains D, then*



$$\left|\begin{matrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ P & Q\end{matrix}\right| = \frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P$$

$$\oint_{\partial D} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A.$$

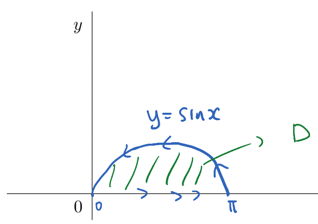
How to remember: The component of dy (i.e. Q) is differentiated w.r.t. x and the component of dx (i.e. P) is differentiated w.r.t. y and multiplied by (-1)

EXAMPLE 6. *Evaluate:*

$$I = \oint_C \underbrace{e^x(1-\cos y)}_{P}\,\mathrm{d}x \underbrace{- e^x(1-\sin y)}_{Q}\,\mathrm{d}y$$

(positively oriented)

where C is the boundary of the domain $D = \{(x,y) : 0 \le x \le \pi, 0 \le y \le \sin x\}$.



Way 1: Parametrize C ☹

Way 2: Green's theorem

$P = e^x(1-\cos y)$

$Q = -e^x(1-\sin y)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^x(1-\sin y) - e^x\cdot\sin y = -e^x + \cancel{e^x \sin y} - \cancel{e^x \sin y} = -e^x$$

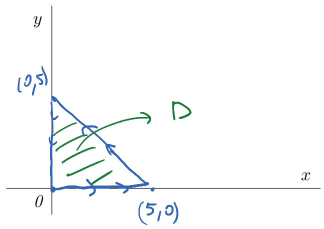
$$I \underset{\text{Green's Thm}}{=} \iint_D (-e^x)\,dx\,dy = -\int_0^{\pi}\left(\int_0^{\sin x} e^x\,dy\right)dx = -\int_0^{\pi} e^x \sin x\,dx =$$

Calc 1 Integration by parts using loops...

$$= \boxed{-\frac{1}{2}\left(e^{\pi}+1\right)}$$

EXAMPLE 7. *Let C be a triangular curve consisting of the line segments from $(0,0)$ to $(5,0)$, from $(5,0)$ to $(0,5)$, and from $(0,5)$ to $(0,0)$. Evaluate the following integral:*

$$I_1 = \oint_C \underbrace{(x^2y+\frac{1}{2}y^2)}_{P}\,\mathrm{d}x + \underbrace{(xy+\frac{1}{3}x^3+3x)}_{Q}\,\mathrm{d}y$$



Use Green's Thm

$P = x^2y + \frac{1}{2}y^2$

$Q = xy + \frac{1}{3}x^3 + 3x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \underbrace{\cancel{y}+\cancel{x^2}+3}_{\frac{\partial Q}{\partial x}} - (\cancel{x^2}+\cancel{y}) = 3$$

$$I_1 \underset{\text{Green's Thm}}{=} \iint_D 3\,dx\,dy = 3\iint_D dx\,dy = 3\cdot \text{Area}(D) = 3\cdot\frac{1}{2}\cdot 5\cdot 5 = \boxed{\frac{75}{2}}$$

Rem: $\text{Area}(D) = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint x\,dy - y\,dx$

SUMMARY: Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$ be a vector field on an open simply connected (no holes) domain D. Suppose that P and Q have continuous partial derivatives through D. Then the facts below are equivalent.

for arbitrary domain D (also with hole)

The field $\mathbf{F}$ is **conservative** on D $\iff$ There exists f s.t. $\nabla f = \mathbf{F}$

(see Corollary 3 above)

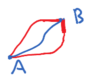
The field $\mathbf{F}$ is **conservative** on D $\iff$ $\int_{\breve{AB}} \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ is independent of path in D

Sketch $\Leftarrow$: WLOG D is connected

Fix a point $A \in D$

Set

$f(B) := \int_{AB} \vec{F}\cdot d\vec{r}$



Claim: $\nabla f = \vec{F}$

always true $\Rightarrow$

$Q_x - P_y = 0$

Under assumption D is simply connected

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ throughout D

$\Leftarrow$ if D is simply connected

$(\Rightarrow)$ $\vec{F} = \langle P, Q\rangle$ is conservative $\Rightarrow$ there exist f s.t. $\begin{cases} f_x = P \xRightarrow{\text{differentiate w.r.t. } y} f_{xy} = P_y \\ f_y = Q \xRightarrow{\text{differentiate w.r.t. } x} f_{yx} = Q_x \end{cases} \Rightarrow P_y = Q_x$

The field $\mathbf{F}$ is **conservative** on D $\iff$ $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = 0$ for every closed curve C in D

Rem

If D is not simply connected $\Rightarrow$ $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ in general does not imply that $\vec{F} = \langle P, Q\rangle$ is conservative

See Example 6 of lecture notes 16.2

EXAMPLE 8. *Determine whether or not the vector field is conservative:*

(a) $\mathbf{F}(x,y) = \langle x^2+y^2, 2xy\rangle$.

$\mathbb{R}^2$ is simply connected

$P = x^2+y^2$, $Q = 2xy$ $\qquad P_y \overset{?}{=} Q_x$ $\qquad P_y = 2y$, $Q_x = 2y$ $\qquad$ yes $P_y = Q_x \Rightarrow \vec{F}$ is conservative

(b) $\mathbf{F}(x,y) = \langle x^2+3y^2+2, 3x+ye^y\rangle$

$P = x^2+3y^2+2$ $\qquad P_y = 6y$

$Q = 3x+ye^y$ $\qquad Q_x = 3$

$6y \not\equiv 3 \Rightarrow \vec{F}$ is not conservative

EXAMPLE 9. *Given* $\mathbf{F}(x,y) = \sin y\,\mathbf{i} + (x\cos y + \sin y)\mathbf{j}$.

(a) *Show that* $\mathbf{F}$ *is conservative.*

$P = \sin y$ $\qquad P_y = \cos y$

$Q = x\cos y + \sin y$ $\qquad Q_x = \cos y$

$P_y \equiv Q_x \Rightarrow F$ is conservative

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$

If f is a potential then $\nabla f = \vec{F} = \langle P, Q\rangle$ $\qquad \langle f_x, f_y\rangle$

$$\begin{cases} f_x = P \\ f_y = Q \end{cases} \Leftrightarrow \begin{cases} f_x = \sin y & (\text{Eq } 1) \\ f_y = x\cos y + \sin y & (\text{Eq } 2) \end{cases}$$

Integrate Eq 1 w.r.t x: $f = \int \sin y \, dx + C(y) =$

$= x\sin y + C(y)$

All terms with x must be cancelled at this step, otherwise there was a mistake before $\leftarrow$ Plug this to (Eq 2): $\frac{\partial}{\partial y}(x\sin y + C(y)) = x\cancel{\cos y} + \underbrace{C'(y)}_{\frac{\partial}{\partial y}C(y)} = x\cancel{\cos y} + \sin y$

$\Rightarrow C'(y) = \sin y \Rightarrow C(y) = \int \sin y + C_1 = -\cos y + C_1 \Rightarrow f(x,y) = x\sin y - \cos y + C_1$

the general form of potential of $\vec{F}$ (we can choose $C_1 = 0 \Rightarrow \boxed{f(x,y) = x\sin y - \cos y}$)

Rem: We can start with (Eq 2) if it is more convenient to integrate Q w.r.t. y than to integrate P w.r.t. x

(c) *Find the work done by the force field* $\mathbf{F}$ *in moving a particle from the point* $(3,0)$ *to the point* $(0,\pi/2)$.

$$\int_{C: (3,0)\to(0,\frac{\pi}{2})} F \cdot d\vec{r} = f\left(0,\frac{\pi}{2}\right) - f(3,0) = \underbrace{0\cdot \sin\frac{\pi}{2} - \cos\frac{\pi}{2}}_{f(0,\frac{\pi}{2})} - \underbrace{(3\cdot\sin 0 - \cos 0)}_{f(3,0)} = 1$$

(d) *Evaluate* $\oint_C \mathbf{F}\,d\mathbf{r}$ *where* C *is an arbitrary* closed *path in* $\mathbb{R}^2$.

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

, because $\vec{F}$ is <u>conservative</u>

$\rightarrow$ Will be given in course of 16.5

EXAMPLE 10. *Given*

$$\mathbf{F} = \langle 2xy^3 + z^2, 3x^2y^2 + 2yz, y^2 + 2xz \rangle .$$

Find a function f s.t. $\nabla f = \mathbf{F}$

1) If such f exists then $\operatorname{curl}\vec{F} = 0$. Check whether it is true.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3+z^2 & 3x^2y^2+2yz & y^2+2xz \end{vmatrix} = (\cancel{2y} - \cancel{2y})\,\hat{i} - (\cancel{2z} - \cancel{2z})\,\hat{j} + (\cancel{6xy^2} - \cancel{6xy^2})\,\hat{k} = 0$$

$\operatorname{curl}\vec{F} = 0$ in $\mathbb{R}^3 \Rightarrow \vec{F}$ is conservative $\Rightarrow$ there exists f s.t. $\nabla f = \vec{F}$

2) Find f by solving the system of equations:

$$\nabla f = \vec{F} \iff \begin{cases} f_x = 2xy^3 + z^2 & (Eq\,1) \\ f_y = 3x^2y^2 + 2yz & (Eq\,2) \\ f_z = y^2 + 2xz & (Eq\,3) \end{cases}$$

$(Eq\,1) \xRightarrow{\text{integrate w.r.t } x} f = \int (2xy^3 + z^2)\,dx = x^2y^3 + z^2x + C(y,z) \quad (*)$

Plug to (Eq 2): $\frac{\partial}{\partial y}(x^2y^3 + z^2x + C(y,z)) = \cancel{3x^2y^2} + \frac{\partial}{\partial y}C(y,z) = \cancel{3x^2y^2} + 2yz$ $\rightarrow$ at this step all terms containing x must be cancelled.

$$\frac{\partial}{\partial y}C(y,z) = 2yz \xRightarrow[\text{w.r.t. } y]{\text{Integrate}} C(y,z) = \int 2yz\,dy = y^2z + C_1(z)$$

Plugging $C(y,z)$ into $(*)$: $f = x^2y^3 + z^2x + y^2z + C_1(z)$

Plug it into (Eq3): $\frac{\partial}{\partial z}(x^2y^3 + z^2x + y^2z + C_1(z)) = \cancel{2zx} + \cancel{y^2} + \frac{d}{dz}C_1(z) = \cancel{y^2} + \cancel{2xz} \Rightarrow \frac{d}{dz}C_1(z) = 0 \Rightarrow C_1(z) \equiv \text{const}$

Finally, $\boxed{f = x^2y^3 + z^2x + y^2z} + C_1$

We can take $C_1 = 0$