F19_LN_16_5

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F19_LN_1...

16.5: Curl and Divergence

Introduce the vector differential operator ∇ as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, R all exist, then the **curl** of $\mathbf{F}$ is the *vector field* on $\mathbb{R}^3$ defined by

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}\hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}\hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\hat{k} =$$

$$\left(\frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q\right)\hat{i} - \left(\frac{\partial}{\partial x}R - \frac{\partial}{\partial z}P\right)\hat{j} + \left(\underbrace{\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P}\right)\hat{k}$$

It is exactly equal to the expression in Green's theorem

EXAMPLE 1. *Find the curl of the vector field*

$$\mathbf{F}(x,y,z) = \langle xy, x^2, yz \rangle.$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & yz \end{vmatrix} = \Big(\underbrace{\frac{\partial}{\partial y}(yz)}_{=z} - \underbrace{\frac{\partial}{\partial z}(x^2)}_{=0}\Big)\hat{i} - \Big(\underbrace{\frac{\partial}{\partial x}(yz)}_{=0} - \underbrace{\frac{\partial}{\partial z}(xy)}_{=0}\Big)\hat{j} +$$

$$+\Big(\underbrace{\frac{\partial}{\partial x}(x^2)}_{=2x} - \underbrace{\frac{\partial}{\partial y}(xy)}_{x}\Big)\hat{k} = \boxed{z\,\hat{i} + x\,\hat{k}} = \boxed{\langle z, 0, x\rangle}$$

Question What is the curl of a two-dimensional vector field

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}\ ?$$

$$\vec{F} = \langle P, Q, 0 \rangle = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j} + 0\cdot\hat{k}$$

Answer:

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \Big(\underbrace{\frac{\partial}{\partial y}(0)}_{0} - \underbrace{\frac{\partial}{\partial z}Q(x,y)}_{0}\Big)\hat{i} - \Big(\underbrace{\frac{\partial}{\partial x}(0)}_{=0} - \underbrace{\frac{\partial}{\partial z}P(x,y)}_{=0}\Big)\hat{j} +$$

$$+\left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\hat{k} = (Q_x - P_y)\,\hat{k}$$

CONCLUSION: Green's Theorem in vector form:

$$\int_{\partial D} P\,dx + Q\,dy = \oint_{\partial D}\mathbf{F}\cdot d\mathbf{r} \underset{\text{Green's Thm}}{=} \iint_D (Q_x - P_y)\,dA = \iint_D (Q_x - P_y)\cdot \underbrace{1}_{\hat{k}\cdot\hat{k}\ \text{dot}}\,dA =$$

$$= \iint_D \underbrace{(Q_x - P_y)\hat{k}}_{\text{curl}\,\vec{F}}\cdot\hat{k}\,dA = \iint_D \text{curl}\,F\cdot\underbrace{\hat{k}}_{\text{normal to } xy\text{-plane}}\,dA$$

THEOREM 2. *If a function $f(x,y,z)$ has continuous partial derivatives of second order then*

$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

Proof. $\operatorname{curl}\nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x}f & \frac{\partial}{\partial y}f & \frac{\partial}{\partial z}f \end{vmatrix} = \underbrace{\left(\frac{\partial}{\partial y}\frac{\partial}{\partial z}f - \frac{\partial}{\partial z}\frac{\partial}{\partial y}f\right)}_{f_{zy}-f_{yz}=0}\hat{i} - (\underbrace{f_{zx}-f_{xz}}_{=0})\hat{j} + (\underbrace{f_{yx}-f_{xy}}_{=0})\hat{k} = 0$

Formally $\nabla\times\nabla f = (\nabla\times\nabla)f = 0$

$\vec{F}$ is conservative in $\mathbb{R}^3$ $\Leftrightarrow$ $\operatorname{curl}\vec{F}=0$

COROLLARY 3. *If* $\mathbf{F}$ *is conservative, then* $\operatorname{curl}\mathbf{F} = \mathbf{0}$. $\Leftrightarrow$ $\begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$

Indeed, if $\vec{F}$ is conservative, then there exists a function f s.t. $\nabla f = \vec{F} \Rightarrow \operatorname{curl}\vec{F} = \operatorname{curl}(\nabla f) = 0$

The proof of the Theorem below requires Stokes' Theorem (Section 14.8).

THEOREM 4. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ (or in a simply connected domain D in $\mathbb{R}^3$) *whose component functions have continuous partial derivatives and* $\operatorname{curl}\mathbf{F} = 0$*, then* $\mathbf{F}$ *is a conservative vector field.*

(one can continuously deform any closed curve into a point, remaining all the time in D)

EXAMPLE 5. *Let* $\mathbf{F}(x,y,z) = \langle x^9, y^9, z^9\rangle$.

(a) *Show that* $\mathbf{F}$ *is conservative.* By Theorem 4 For this it is enough to check that $\operatorname{curl}\vec{F} \equiv 0$

$$\operatorname{curl} F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^9 & y^9 & z^9 \end{vmatrix} = \left(\underbrace{\frac{\partial}{\partial y}z^9}_{=0} - \underbrace{\frac{\partial}{\partial z}y^9}_{=0}\right)\hat{i} - \left(\underbrace{\frac{\partial}{\partial x}z^9}_{=0} - \underbrace{\frac{\partial}{\partial z}x^9}_{=0}\right)\hat{j} + \left(\underbrace{\frac{\partial}{\partial x}y^9}_{=0} - \underbrace{\frac{\partial}{\partial y}x^9}_{=0}\right)\hat{k} = 0 \Rightarrow \vec{F} \text{ is conservative}$$

(b) *Find a function* f *s.t.* $\nabla f = \mathbf{F}$.

Way one: Solve the corresp. system (as in Example 10 of 16.3-4)

Way two: Guess $f = \frac{1}{10}(x^{10} + y^{10} + z^{10})$

Rem. If $\vec{F} = (P(x), Q(y), R(z))$ then it is always conservative & the potential $f = U(x) + V(y) + W(z)$ where U, V, W are antiderivatives (in a sense of Calc1) of P, Q & R, resp.

(c) *Evaluate* $\displaystyle\int_{(1,0,1)}^{(-1,-1,-1)} \mathbf{F}\cdot d\mathbf{r}$. $= f(-1,-1,-1) - f(1,0,1) = \frac{1}{10}(1+1+1) - \frac{1}{10}(1+0+1) = \frac{3}{10} - \frac{2}{10} = \boxed{\frac{1}{10}}$

If $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives P_x, Q_y, R_z exist, then the **divergence of F** is the *scalar field* on defined by

dot product

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R$$

a scalar function / a scalar field

EXAMPLE 6. *Find the divergence of the vector field*

$$\mathbf{F}(x,y,z) = \langle \underbrace{\sin(xyz)}_{P}, \underbrace{x^2}_{Q}, \underbrace{yz}_{R} \rangle .$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}P + \frac{\partial}{\partial y}Q + \frac{\partial}{\partial z}R = \frac{\partial}{\partial x}\sin(xyz) + \frac{\partial}{\partial y}x^2 + \frac{\partial}{\partial z}yz = \boxed{yz\cos(xyz) + y}$$

Previously we discussed curl grad $f = 0$

THEOREM 7. *If the components of a vector field* $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ *has continuous partial derivatives of second order then*

$$\operatorname{div}\operatorname{curl}\mathbf{F} = 0.$$

Recall that for the triple scalar product $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$

Proof. Formally $\operatorname{div}\operatorname{curl}\vec{F} = \nabla \cdot (\nabla \times \vec{F}) = 0$

EXAMPLE 8. *Is there a vector field* **G** *on* $\mathbb{R}^3$ *s.t.* curl $G = \underbrace{\langle yz, xyz, zy \rangle}_{\vec{F}}$?

If $\operatorname{curl}\vec{G} = \vec{F} \Rightarrow 0 = \operatorname{div}\operatorname{curl}\vec{G} = \operatorname{div}\vec{F} \Rightarrow \operatorname{div}\vec{F} = 0$

Take div from both sides

Question Is $\operatorname{div}\vec{F} = 0$?

$$\operatorname{div}\vec{F} = \underbrace{\frac{\partial}{\partial x}(yz)}_{=0} + \underbrace{\frac{\partial}{\partial y}(xyz)}_{=xz} + \underbrace{\frac{\partial}{\partial z}(zy)}_{=y} = xz + y \neq 0 \Rightarrow$$ Answer is no!