F19_LN_16_6

Friday, November 22, 2019 6:53 AM

F19_LN_1...

16.6: Parametric surfaces and their areas

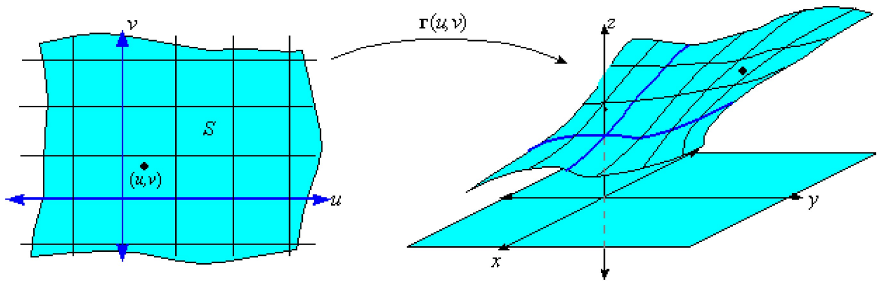
Consider a continuous vector valued function of two variables

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D.$$

Parametric surface:

$$S: \; x = x(u,v), \quad y = y(u,v), \quad z = z(u,v), \quad (u,v) \in D.$$

In other words, the surface S is traces out by the position vector $\mathbf{r}(u,v)$ as (u,v) moves throughout the region D.



EXAMPLE 1. *Determine the surface given by the parametric representation*

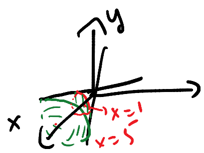
$$\mathbf{r}(u,v) = \langle u, u\cos v, u\sin v \rangle, \quad 1 \le u \le 5, \quad 0 \le v \le 2\pi$$

$$\begin{cases} x = u \\ y = u\cos v \\ z = u\sin v \end{cases}$$

Find the relation between x, y, and z

$y^2 + z^2 = u^2 = x^2 \Rightarrow x^2 = y^2 + z^2$ – a cone along x-axis

$1 \le u \le 5 \iff 1 \le x \le 5$



– a lateral surface of a coffee cup.

EXAMPLE 2. *Give parametric or vector representations for each of the following surfaces:*

(a) *the cylinder:* $x^2 + y^2 = 9, \quad 1 \le z \le 5$.

Use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$x^2+y^2=9 \Rightarrow r^2 = 9 \Rightarrow r = 3$

Our parameters are θ & z

$$\boxed{\begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \\ z = z \end{cases} \quad \begin{array}{l} 0 \le \theta \le 2\pi \\ 1 \le z \le 5 \end{array}}$$

(b) *the upper half-sphere:* $z = \sqrt{100 - x^2 - y^2}$. – upper hemisphere of radius 10 around the origin

Way 1 : Using that this is a graph of x & y $\Rightarrow$ x & y are parameters

$$\boxed{\begin{cases} x = x \\ y = y \\ z = \sqrt{100 - x^2 - y^2} \end{cases} \quad (x,y) \in D = \{(x,y): x^2+y^2 \le 100\}}$$

where (x,y) satisfy $100 - x^2 - y^2 \ge 0 \Leftrightarrow x^2+y^2 \le 100 \Rightarrow$

Way 2 Using spherical coordinates : $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$

$z = \sqrt{100-x^2-y^2} \Leftrightarrow$ $x^2+y^2+z^2 = 100 \Rightarrow \rho^2 = 100 \Leftrightarrow \rho = 10$

$z \ge 0 \Rightarrow \rho\cos\varphi \ge 0 \Leftrightarrow \cos\varphi \ge 0 \Leftrightarrow 0 \le \varphi \le \frac{\pi}{2}$

Our parameters are φ and θ and the parametric equation is :

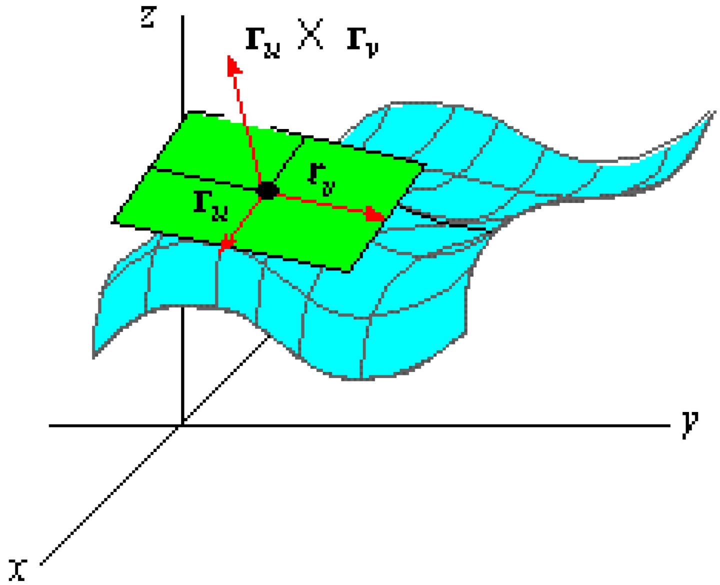
$$\begin{cases} x = 10\sin\varphi\cos\theta \\ y = 10\sin\varphi\sin\theta \\ z = 10\cos\varphi \end{cases} \quad \begin{array}{l} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{array}$$

CONCLUSION: To parametrize surfaces in majority of the exercises that will be given you may use polar, cylindrical or spherical coordinates, or

- $z = f(x,y) \longrightarrow \mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$
- $y = f(x,z) \longrightarrow \mathbf{r}(x,z) = x\mathbf{i} + f(x,z)\mathbf{j} + z\mathbf{k}$
- $x = f(y,z) \longrightarrow \mathbf{r}(y,z) = f(y,z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

- **Tangent planes**:

PROBLEM: Find a normal vector to the tangent plane to a parametric surface S given by a vector function $\mathbf{r}(u,v)$ at a point P_0 with position vector $\mathbf{r}(u_0,v_0)$, i.e. $P_0(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$



$\vec{r}_u = \langle x_u, y_u, z_u \rangle$
$\vec{r}_v = \langle x_v, y_v, z_v \rangle$

The **normal vector**

$$\mathbf{N} = \mathbf{N}(u,v) = \vec{r}_u(u,v) \times \vec{r}_v(u,v)$$

If a normal vector is not $\mathbf{0}$ then the surface S is called **smooth** (it has no "corner").

Special Case: a surface S given by a *graph* $z = f(x,y)$. Then one can choose the following parametrization of S:

$$\mathbf{r}(x,y) = \langle x, y, f(x,y) \rangle \Rightarrow \begin{array}{l} \vec{r}_x = \langle 1, 0, f_x \rangle \\ \vec{r}_y = \langle 0, 1, f_y \rangle \end{array}$$

and the then the normal vector is

$$\mathbf{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x \hat{i} - f_y \hat{j} + \hat{k} =$$

$$= \langle -f_x, -f_y, 1 \rangle$$

EXAMPLE 3. *Find the tangent plane to the surface with parametric equations* $x = uv + 1, y = ue^v, z = ve^u$ *at the point* $(1, 0, 0)$.

$\vec{r} = \langle uv+1, ue^v, ve^u \rangle$

$\vec{r}_u = \langle v, e^v, ve^u \rangle$

$\vec{r}_v = \langle u, ue^v, e^u \rangle$

Find (u,v) that correspond to the point $(1,0,0)$:

$$\begin{cases} 1 = uv+1 \Rightarrow uv = 0 \\ 0 = ue^v \Rightarrow u=0 \\ 0 = ve^u \Rightarrow v=0 \end{cases} \Rightarrow (u,v) = (0,0)$$

$\vec{r}_u(0,0) = \langle 0, 1, 0 \rangle$

$\vec{r}_v(0,0) = \langle 0, 0, 1 \rangle$

$$N(0,0) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{i} = \langle 1, 0, 0 \rangle$$

$\Downarrow$

The equation of the tangent plane is

$1 \cdot (x-1) + 0 \cdot y + 0 \cdot z = 0 \iff x - 1 = 0 \iff x = 1$

- **Surface Area:**

Consider a smooth surface S given by

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

then

$N(u,v) = r_u \times r_v$

$$dS = |N(u, v)|dudv = |r_u \times r_v| du dv$$

and the **surface area**

$$A(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

See notes of section 513 for explanations.

REMARK 4. *Special Case*: a surface S given by a *graph* $z = f(x,y)$ we have

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

and

$$dS = |\mathbf{N}(x,y)|\, dA = |\langle -f_x, -f_y, 1\rangle|\, dx\,dy = \sqrt{1+f_x^2+f_y^2}\, dx\,dy$$

see the end of page 3

EXAMPLE 5. *Find the surface area of the surface*

$$S: \quad x = uv, \quad y = u+v, \quad z = u-v, \qquad u^2+v^2 \le 1.$$

$\vec{r}(u,v) = \langle uv, u+v, u-v\rangle$

$\vec{r}_u = \langle v, 1, 1\rangle$

$\vec{r}_v = \langle u, 1, -1\rangle$

$$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} = (-1-1)\hat{i} - (-v-u)\hat{j} + (v-u)\hat{k} = \langle -2, u+v, v-u\rangle$$

$$|\vec{N}(u,v)| = \sqrt{4+(u+v)^2+(v-u)^2} = \sqrt{4+u^2+\cancel{2uv}+v^2+v^2-\cancel{2uv}+u^2} = \sqrt{4+2u^2+2v^2}$$

$$A(S) = \iint_{u^2+v^2\le 1} |\vec{r}_u \times \vec{r}_v|\, du\,dv = \iint_{u^2+v^2\le 1} \sqrt{4+2(u^2+v^2)}\, du\,dv =$$

the double integral

use polar change $\begin{matrix} u = r\cos\theta \\ v = r\sin\theta \end{matrix} \Rightarrow \begin{matrix} r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4+2r^2}\cdot r\, dr\, d\theta = 2\pi \int_0^1 \sqrt{4+2r^2}\, r\, dr \quad \underline{\text{Calc 1}} \ldots$$

EXAMPLE 6. *Find the surface area of the part paraboloid $z = x^2 + y^2$ between two planes: $z = 0$ and $z = 4$.*

Way 1 It is a special case

$z = f(x,y) = x^2+y^2, \quad x^2+y^2 \le 4$

$\overrightarrow{N(x,y)} = |\vec{r}_x \times \vec{r}_y| = \sqrt{1+f_x^2+f_y^2} = \sqrt{(2x)^2+(2y)^2+1} = \sqrt{4(x^2+y^2)+1}$

$A(S) = \iint_S dS = \iint_{x^2+y^2\le 4} \sqrt{1+f_x^2+f_y^2}\,dx\,dy = \iint_{x^2+y^2\le 4} \sqrt{4(x^2+y^2)+1}\,dx\,dy \underset{\text{polar change}}{=} \int_0^{2\pi} d\theta \int_0^2 \sqrt{4r^2+1}\, r\,dr =$

$2\pi\int_0^2 \sqrt{4r^2+1}\,r\,dr \underset{\text{u-substitution}}{=} \ldots$ Calc 1

Way 2 Use cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$z = x^2+y^2 \Rightarrow z = r^2$

$0 \le z \le 4 \Leftrightarrow 0 \le r^2 \le 4 \Leftrightarrow 0 \le r \le 2$

$\Rightarrow$ the parametrization of the surface is $x = r\cos\theta$, $y = r\sin\theta$, $z = r^2$, $0\le r\le 2$, $0 \le \theta \le 2\pi$ or

$\vec{\varphi}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$

$\vec{\varphi}_r = \langle \cos\theta, \sin\theta, 2r\rangle$

$\vec{\varphi}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$\vec{N}(r,\theta) = \vec{\varphi}_r \times \vec{\varphi}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} =$

$= -2r^2\cos\theta\,\hat{i} + 2r^2\sin\theta\,\hat{j} + r(\underbrace{\cos^2\theta+\sin^2\theta}_{1})\hat{k}$

$= \langle -2r^2\cos\theta, 2r^2\sin\theta, r\rangle$

$\Rightarrow |\vec{N}(r,\theta)| = \sqrt{\underbrace{4r^4\cos^2\theta + 4r^4\sin^2\theta}_{4r^4} + r^2} =$

$= \sqrt{4r^4+r^2} = \sqrt{r^2}\sqrt{4r^2+1} = r\sqrt{4r^2+1}$

$\Rightarrow A(S) = \iint_S dS = \int_0^{2\pi}\int_0^2 |\vec{N}(r,\theta)|\,dr\,d\theta =$

$= \int_0^{2\pi} d\theta \int_0^2 r\sqrt{4r^2+1}\,dr = 2\pi\int_0^2 r\sqrt{4r^2+1}\,dr$

$= \ldots$ Calc 1 u-substitution