F19_LN_16_6

Friday, November 22, 2019 6:53 AM

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16.6: Parametric surfaces and their areas

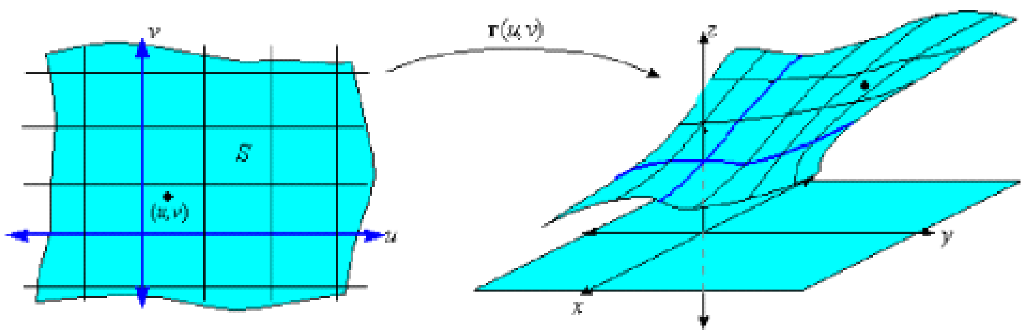
Consider a continuous vector valued function of two variables

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D.$$

Parametric surface:

$$S: \; x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in D.$$

In other words, the surface S is traces out by the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D.



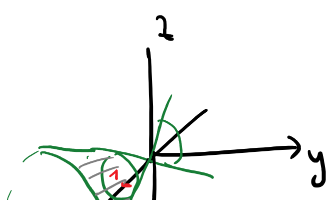
EXAMPLE 1. *Determine the surface given by the parametric representation*

$$\mathbf{r}(u, v) = \langle u, u\cos v, u\sin v\rangle, \quad 1 \le u \le 5, \quad 0 \le v \le 2\pi$$

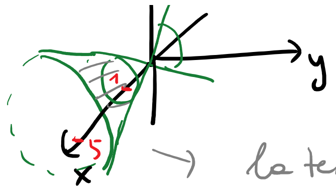
$$\begin{cases} x = u \\ y = u\cos v \\ z = u \sin v \end{cases} \quad y^2 + z^2 = u^2 = x^2 \implies \underline{x^2 = y^2 + z^2}$$

a cone symmetric w.r.t. x-axis

$1 \le u \le 5 \iff 1 \le x \le 5$



$$S = \left\{ (x, y, z) : \begin{array}{l} x^2 = y^2 + z^2 \\ 1 \le x \le 5 \end{array} \right.$$



$S = \{(x,y,z):\ x^2 = y^2 + z^2,\ 1 \le x \le 5\}$ $\to$ lateral surface of a cup.

EXAMPLE 2. *Give parametric or vector representations for each of the following surfaces:*

(a) *the cylinder:* $x^2 + y^2 = 9, \quad 1 \le z \le 5.$

Use cylindrical coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$x^2+y^2=9 \Rightarrow r^2 = 9 \Leftrightarrow r = 3$

Our parameters are θ & z

The parametrization is $\boxed{\begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \\ z = z \end{cases} \quad \begin{matrix} 1 \le z \le 5 \\ 0 \le \theta \le 2\pi \end{matrix}}$

(b) *the upper half-sphere:* $z = \sqrt{100 - x^2 - y^2}$. $\Leftrightarrow$ $x^2+y^2+z^2 = 100$, $z \ge 0$

Way 1 Our surface is a graph of a function $f(x,y) = \sqrt{100-x^2-y^2}$ where $x^2+y^2 \le 100$. We can use x & y as the parameters:

$$\begin{cases} x = x \\ y = y \\ z = \sqrt{100-x^2-y^2} \end{cases} \qquad \vec{r}(x,y) = \langle x, y, \sqrt{100-x^2-y^2} \rangle \text{ where } x^2+y^2 \le 100$$

Way 2 Using spherical coordinates

$z = \sqrt{100-x^2-y^2} \Leftrightarrow \{x^2+y^2+z^2 = 100 \Rightarrow \rho^2 = 100 \Leftrightarrow \rho = 10$

$x = \rho \sin\varphi \cos\theta$

Way 2 Using spherical coordinates

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$z = \sqrt{100 - x^2 - y^2} \iff \begin{cases} x^2+y^2+z^2 = 100 \Rightarrow \rho^2 = 100 \iff \rho = 10 \\ z \geq 0 \iff \rho\cos\varphi \geq 0 \iff \cos\varphi \geq 0 \iff 0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

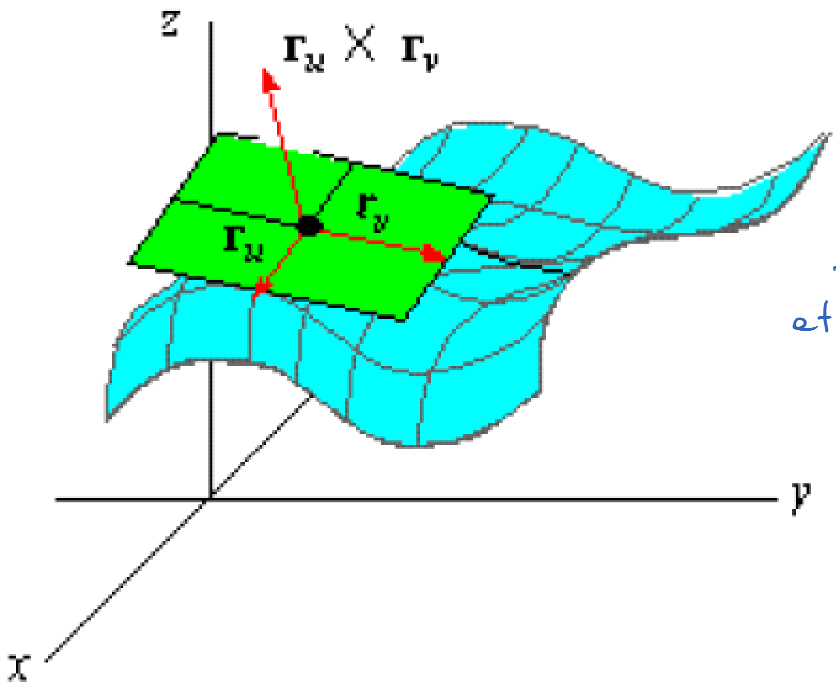
Our parameters are φ & θ:
$$\begin{cases} x = 10 \sin\varphi \cos\theta \\ y = 10 \sin\varphi \sin\theta \\ z = 10 \cos\varphi \end{cases} \qquad \begin{aligned} 0 \leq \varphi \leq \frac{\pi}{2} \\ 0 \leq \theta \leq 2\pi \end{aligned}$$

CONCLUSION: To parametrize surfaces in majority of the exercises that will be given you may use polar, cylindrical or spherical coordinates, or

- $z = f(x, y) \longrightarrow \mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$
- $y = f(x, z) \longrightarrow \mathbf{r}(x, z) = x\mathbf{i} + f(x, z)\mathbf{j} + z\mathbf{k}$
- $x = f(y, z) \longrightarrow \mathbf{r}(y, z) = f(y, z)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

- **Tangent planes:**

PROBLEM: Find a normal vector to the tangent plane to a parametric surface S given by a vector function $\mathbf{r}(u, v)$ at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$, i.e. $P_0(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$



$\vec{r}_u = \langle x_u, y_u, z_u \rangle$
$\vec{r}_v = \langle x_v, y_v, z_v \rangle$
$\downarrow$
both vectors are tangent to surface at the considered point

x ✓

The **normal vector**

$$\mathbf{N} = \mathbf{N}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

If a normal vector is not **0** then the surface S is called **smooth** (it has no "corner").

Special Case: a surface S given by a *graph* $z = f(x,y)$. Then one can choose the following parametrization of S:

$$\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle \Rightarrow \begin{array}{l} \vec{r}_x = \langle 1, 0, f_x\rangle \\ \vec{r}_y = \langle 0, 1, f_y\rangle \end{array}$$

and the then the normal vector is

$$\mathbf{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\hat{i} - f_y\hat{j} + \hat{k} = \langle -f_x, -f_y, 1\rangle$$

EXAMPLE 3. *Find the tangent plane to the surface with parametric equations $x = uv + 1, y = ue^v, z = ve^u$ at the point $(1, 0, 0)$.*

$\vec{r} = \langle uv+1, ue^v, ve^u\rangle$

$\vec{r}_u = \langle v, e^v, ve^u\rangle$

$\vec{r}_v = \langle u, ue^v, e^u\rangle$

↓ plug $(u,v) = (0,0)$

Find (u,v) corresponding to $(x,y,z) = (1,0,0)$ by solving:

$$\begin{cases} 1 = uv+1 \\ 0 = ue^v \Rightarrow u = 0 \\ 0 = ve^u \Rightarrow v = 0 \end{cases} \Rightarrow u = v = 0$$

$$\begin{array}{l} \vec{r}_u = \langle 0, 1, 0\rangle \\ \vec{r}_v = \langle 0, 0, 1\rangle \end{array} \Rightarrow \vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \hat{i} = \langle 1, 0, 0\rangle$$

The equation of the tangent plane is

$$1\cdot(x-1) + 0\cdot y + 0\cdot z = 0 \iff x - 1 = 0 \iff \boxed{x = 1}$$

$$1\cdot(x-1)+0\cdot y+0\cdot z=0 \Leftrightarrow x-1=0 \Leftrightarrow \boxed{x=1}$$

- **Surface Area:**

Consider a smooth surface S given by

$$\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}, \quad (u,v)\in D,$$

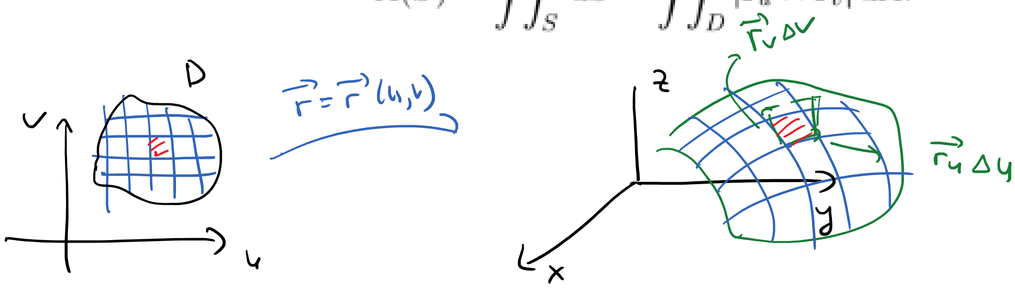
$$\vec{N}(u,v)=\vec{r}_u(u,v)\times\vec{r}_v(u,v)$$

then

$$\mathrm{d}S=|N(u,v)|\mathrm{d}u\mathrm{d}v=|\vec{r}_u(u,v)\times\vec{r}_u(u,v)|\,du\,dv$$

and the **surface area**

$$A(S)=\iint_S \mathrm{d}S=\iint_D |\mathbf{r}_u\times\mathbf{r}_v|\,\mathrm{d}A.$$



$dS \approx$ the area of the parallelogram generated by the vector $\vec{r}_u\Delta u$ & $\vec{r}_v\Delta v$ $=|\vec{r}_u\Delta u\times\vec{r}_u\Delta v|$ $=|\vec{r}_u\times\vec{r}_v|\underset{du}{\Delta u}\,\underset{dv}{\Delta v}$

REMARK 4. *Special Case:* a surface S given by a *graph* $z=f(x,y)$ we have

$$\mathbf{r}(x,y)=x\mathbf{i}+y\mathbf{j}+f(x,y)\mathbf{k}$$

and

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$$\mathrm{d}S=|\mathbf{N}(x,y)|\,\mathrm{d}A=|\langle -f_x,-f_y,1\rangle|\,dx\,dy=\sqrt{1+f_x^2+f_y^2}\,dx\,dy$$

EXAMPLE 5. *Find the surface area of the surface*

EXAMPLE 5. *Find the surface area of the surface*

$$S: \quad x = uv, \quad y = u + v, \quad z = u - v, \qquad u^2 + v^2 \le 1.$$

$\vec{r}(u,v) = \langle uv, u+v, u-v \rangle$, $\quad D = \{(u,v): u^2+v^2 \le 1\}$

$\vec{r}_u = \langle v, 1, 1 \rangle$

$\vec{r}_v = \langle u, 1, -1 \rangle$

$\vec{N}(u,v) = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} =$

$= (-1-1)\hat{i} - (-v-u)\hat{j} + (v-u)\hat{k} = -2\hat{i} + (u+v)\hat{j} + (v-u)\hat{k}$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-2)^2 + (u+v)^2 + (v-u)^2} = \sqrt{4 + u^2 + \cancel{2uv} + v^2 + v^2 - \cancel{2uv} + u^2} = \sqrt{4 + 2u^2 + 2v^2}$

$\Rightarrow A(S) = \iint_S dS = \iint_D \sqrt{4 + 2u^2 + 2v^2}\, du\, dv =$ Change to polar $u = r\cos\theta$, $v = r\sin\theta$ $\int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2} \cdot r\, dr\, d\theta =$

$= 2\pi \int_0^1 \sqrt{4 + 2r^2} \cdot r\, dr\, d\theta =$ Calc 1 u-substitution

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EXAMPLE 6. *Find the surface area of the part paraboloid* $z = x^2 + y^2$ *between two planes:* $z = 0$ *and* $z = 4$.

See completed notes of section 501 for the solution

See completed notes of section [illegible]

for the solution.