F19_LN_16_7

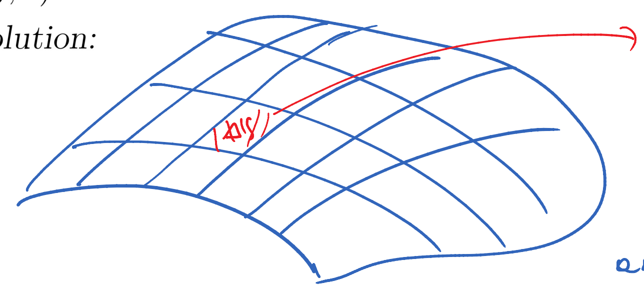
Sunday, November 24, 2019   10:55 PM

F19_LN_1...

16.7: Surface Integrals

Problem: Find the **mass** of a thin sheet (say, of aluminum foil) which has a shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$.

Solution:



$m = \rho(x^*, y^*, z^*)\,\Delta S$

summing up masses of all pieces and making partitions finer and finer we get

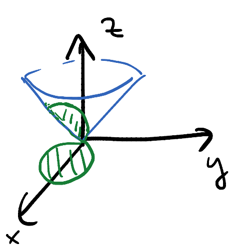
$$m(S) = \iint_S \rho\, dS$$

If S is given by $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$, $(u, v) \in D$, then the **surface integral of f over the surface** S is:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\underbrace{|\mathbf{N}(u, v)|\, dA}_{dS} = \iint_D f(\vec{r}(u,v))\underbrace{|\vec{r}_u \times \vec{r}_v|\, du\, dv}_{dS}$$

EXAMPLE 1. *Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $\underbrace{x^2+y^2 \le 2x}$, if its density is a function $\rho(x, y, z) = x^2+y^2+z^2$.*

$(x-1)^2 + y^2 \le 1 \to$ a unit disk aroun $(1,0)$



Use the special parametrization of the cone as the graph of function $f(x,y) = \sqrt{x^2+y^2}$

$\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle$, $(x,y) \in D = \{(x,y) : x^2+y^2 \le 2x\}$

$$\vec r(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle, \quad (x,y)\in D = \{(x,y): x^2+y^2\le 2x\}$$

In this special case

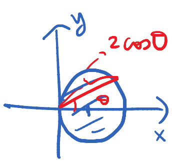
$$|\vec N(x,y)| = |\underbrace{\vec r_x \times \vec r_y}_{\langle -f_x,-f_y,1\rangle}| = \sqrt{1+f_x^2+f_y^2} = \sqrt{1+\Big(\underbrace{\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}}_{f_x}\Big)^2+\Big(\underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{f_y}\Big)^2} =$$

$$= \sqrt{1+\underbrace{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}_{\frac{x^2+y^2}{x^2+y^2}=1}} = \sqrt{2}$$

$$m = \iint_S \rho(x,y,\underset{\sqrt{x^2+y^2}}{\underset{\parallel}{z}})\,dS = \iint_D \Big(x^2+y^2+\overbrace{\big(\underbrace{\sqrt{x^2+y^2}}_{z^2}\big)^2}^{x^2+y^2}\Big)\underbrace{\sqrt{2}\,dx\,dy}_{|\vec N(x,y)|\,dx\,dy} =$$

$$= \iint_D 2(x^2+y^2)\sqrt{2}\,dx\,dy \underset{\text{polar change}}{=} 2\sqrt{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \overbrace{r^2\cdot r}^{r^3}\,dr\,d\theta = 2\sqrt{2}\cdot\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^4}{4}\Big|_{r=0}^{2\cos\theta} d\theta =$$

$x = r\cos\theta$
$y = r\sin\theta$



$D = \{(x,y): x^2+y^2 \le 2x\}$

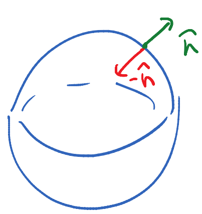
In r: $D^* = \{(r,\theta): r^2 \le 2r\cos\theta \iff 0 \le r \le 2\cos\theta,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$

$$= \frac{\sqrt{2}}{2}\int_{-\pi/2}^{\pi/2} 2^4\cos^4\theta\,d\theta = \ldots$$

Calc 1 + trigonometric ... $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

• **Oriented surfaces.** We consider only two-sided surfaces.

Let a surface S has a tangent plane at every point (except at any boundary points). There are two unit normal vectors at (x,y,z): $\hat{\mathbf{n}}$ and $-\hat{\mathbf{n}}$.



If it is possible to choose a unit normal vector $\hat{\mathbf{n}}$ at every point (x,y,z) of a surface S so that $\hat{\mathbf{n}}$ varies continuously over S, then S is called **oriented surface** and the given choice of $\hat{\mathbf{n}}$ provides S with an **orientation**. There are two possible orientations for any orientable surface:

and the given choice of $\hat{\mathbf{n}}$ provides S with an **orientation**. There are two possible orientations for any orientable surface:

Convention: *For closed surfaces (no boundary) the positive orientation is outward.*

- **Surface integrals of vector fields.**

DEFINITION 2. *If* $\mathbf{F}$ *is a continuous vector field defined on an oriented surface* S *with unit normal vector* $\hat{\mathbf{n}}$, *then the* **surface integral of F over** S *is*

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \underbrace{\mathbf{F}\overset{\text{dot product}}{\cdot}\hat{\mathbf{n}}}_{\text{a scalar function}}\, dS.$$

This integral is also called the **flux** *of* $\mathbf{F}$ *across* S.

If F represent the velocity of the motion of the liquid ↓ flux represent the volume of the liquid passing through S at the unit of time in the direction of $\hat{n}$.

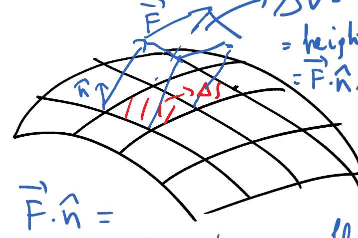
Note that if S is given by $\mathbf{r}(u,v)$, $(u,v)\in D$, then

$$\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{\vec{N}(u,v)}{|\vec{N}(u,v)|} = \pm\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$$

and

$$\underbrace{d\mathbf{S}}_{d\vec{S}} = \pm\hat{n}\,\underbrace{dS}_{|\vec{r}_u\times\vec{r}_v|\,du\,dv} = \pm\frac{\vec{r}_u\times\vec{r}_v}{\cancel{|\vec{r}_u\times\vec{r}_v|}}\cdot\cancel{|\vec{r}_u\times\vec{r}_v|}\,du\,dv = \pm(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

the sign is chosen according to the initial orientation on S



$\Delta V = \text{height}\cdot\Delta S = \vec{F}\cdot\hat{n}\,\Delta S$

$\vec{F}\cdot\hat{n}$ = the hight of a parallelepiped with the basis ΔS

according to the initial orientation on S

The hight of a parallelepiped with the basis ΔS

Finally,

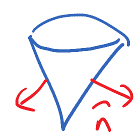
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \pm\iint_D \underbrace{\vec F(\vec r(u,v))\cdot(\vec r_u\times\vec r_v)}_{\text{the triple scalar product}}\,du\,dv =$$

$\vec F = \langle P, Q, R\rangle$

$$= \pm\iint_D \begin{vmatrix} P(\vec r(u,v)) & Q(\vec r(u,v)) & R(\vec r(u,v))\\ x_u & y_u & z_u \\ x_v & y_v & z_v\end{vmatrix}\, du\,dv$$

EXAMPLE 3. *Find the flux of the vector field*

$$\mathbf{F} = \langle x^2, y^2, z^2\rangle$$



→ downward

$\hat n$

across the surface

$$S = \{z^2 = x^2+y^2, 0\le z\le 2\}.$$ with orientation out of S

$z^2 = x^2+y^2$, $0 \le z \le 2$ $\Rightarrow$ $z = \sqrt{x^2+y^2}$ $\Rightarrow$ the parametrization

$0 \le x^2+y^2 \le 4$

is special (by x & y) $\qquad \vec r(x,y) = \langle x, y, \underbrace{\sqrt{x^2+y^2}}_{= z}\rangle$, $\underbrace{x^2+y^2\le 4}_{D}$

$\Downarrow$

$$\vec N(x,y) = \pm\vec r_x\times\vec r_y = \pm\begin{vmatrix} \hat i & \hat j & \hat k\\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}}\\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}}\end{vmatrix} = \pm\left\langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right\rangle$$

$-$: In order to be downward

1: upward

$$\text{Flux} = \iint_S \vec F\cdot\hat n\, dS = \iint \langle x^2, y^2, x^2+y^2\rangle\cdot\left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right\rangle dx\,dy$$

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_D \left\langle x^2, y^2, \underbrace{x^2+y^2}_{z^2} \right\rangle \cdot \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle dx\,dy$$

to be downward

$$= \iint_D \left(\frac{x^3+y^3}{\sqrt{x^2+y^2}} - (x^2+y^2) \right) dx\,dy =$$

polar change $x = r\cos\theta$, $y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^{2} \left(\frac{r^3\cos^3\theta + r^3\sin^3\theta}{r} - r^2 \right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} \left(r^2\cos^3\theta + r^2\sin^3\theta - r^3 \right) dr$$

$$\int_0^{2\pi} \cos^3\theta = \int_0^{2\pi} \sin^3\theta = 0$$

$$= -\int_0^{2\pi}\int_0^{2} r^3\,dr = -2\pi \cdot \frac{2^4}{4} = \boxed{-8\pi}$$