F19_LN_16_8

Tuesday, December 3, 2019 7:02 PM

F19_LN_1...

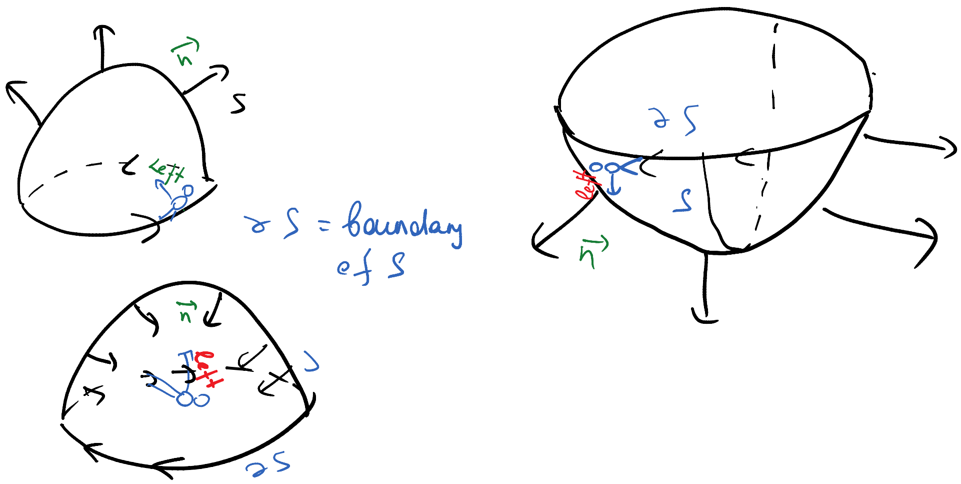
16.8: STOKES' THEOREM

Stokes' Theorem can be regarded as a 3-dimensional version of Green's Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\, dA.$$

the normal to the plane $z=0$, that contains D

Let S be an oriented surface with unit normal vector $\hat{\mathbf{n}}$ and with the boundary curve C (which is a space curve).



The orientation on S induces the **positive orientation of the boundary curve** C: if you walk in the positive direction around C with your head pointing in the direction of $\hat{\mathbf{n}}$, then the surface will always be on your left.

The positively oriented boundary curve of an oriented surface S is often written as ∂S.

Stokes' Theorem: *Let S be an oriented piece-wise-smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

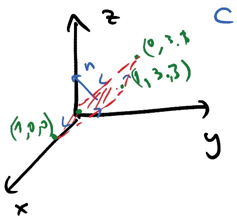
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S},$$

or

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \oint_{\partial S} \mathbf{F}\cdot d\mathbf{r}.$$

EXAMPLE 1. *Find the work performed by the forced field* $\mathbf{F}(x,y,z) = \langle 3x^8, 4xy^3, y^2x\rangle$ *on a particle that traverses the curve* C *in the plane* $z = y$ *consisting of 4 line segments from* $(0,0,0)$ *to* $(1,0,0)$*, from* $(1,0,0)$ *to* $(1,3,3)$*, from* $(1,3,3)$ *to* $(0,3,3)$*, and from* $(0,3,3)$ *to* $(0,0,0)$.

$W = \oint_C \vec{F}\cdot d\vec{r}$ **Way 1** (direct calculation) ☹ Parametrize 4 segments that form the curve C and calculate four integrals ☹



Way 2 (using the Stokes theorem) ☺ C is a closed curve so we can apply Stokes theorem. For this, choose the surface S as a part of the plane $z=y$ bounded by C (actually this is a rectangle)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^3 & 4xy^3 & y^2x \end{vmatrix} = (2yx-0)\hat{i} - (y^2-0)\hat{j} + 4y^3\hat{k} =$$

$$= \langle 2yx, y^2, 4y^3\rangle$$

Parametrize S (special case, i.e of the graph $z=f(x,y)=y$)

$\vec{r}(x,y) = \langle x,y,y\rangle$, $D=\{(x,y): 0\le x\le 1, 0\le y\le 3\}$ → the projection to (x,y)-plane

The normal is $\pm\langle -f_x, -f_y, 1\rangle$. We choose $+$ because in order that the given orientation on C will be positive, the positive normal on S must be upward. ⇒ $\vec{N}(x,y) = \langle -f_x, -f_y, 1\rangle$ where $f(x,y)=y$ ⇒

$\vec{N}(x,y) = \langle 0,-1,1\rangle$.

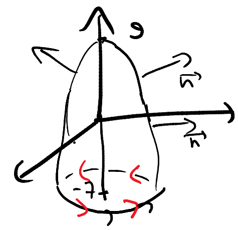
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \operatorname{curl}\vec{F}(x,y)\cdot \underbrace{\vec{N}(x,y)}_{\vec{r}_x\times\vec{r}_y}\,dx\,dy =$$

$$= \int_0^3\int_0^1 \langle 2yx, -y^2, 4y^3\rangle\cdot\langle 0,-1,1\rangle\,dx\,dy = \int_0^1\left(\int_0^3 (y^2+4y^3)\,dy\right)dx =$$

$$= \underbrace{\int_0^1 dx}_{1}\int_0^3 (y^2+4y^3)\,dy = \frac{3^3}{3} + \cancel{4}\cdot\frac{3^4}{\cancel{4}} = 9+81 = \boxed{90}.$$

EXAMPLE 2. *Verify Stokes' Theorem* $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S}\vec{F}\cdot d\vec{r}$ *for the vector field* $\vec{F} = \langle 3y, 4z, -6x\rangle$ *and the paraboloid* $z = 9 - x^2 - y^2$ *that lies above the plane* $z = -7$ *and oriented upward. Be sure to check and explain the orientations.*

Solution: Use the following steps:



•Parametrize the boundary circle ∂S and compute the line integral.

∂S is the line of intersection of the paraboloid

$z = 9 - x^2 - y^2$ and the plane $z=-7$ $\Rightarrow -7 = 9 - x^2 - y^2 \Rightarrow$

$\boxed{x^2+y^2 = 16, \quad z=-7}$ $\Rightarrow$ Parametrization of ∂S:

$x = 4\cos\theta,\ y = 4\sin\theta,\ z = -7,\ 0\le\theta\le 2\pi \Rightarrow r'(\theta) = \langle -4\sin\theta, 4\cos\theta, 0\rangle$

The orientation is counter-clockwise if seen from above.

$$\oint_{\partial S}\vec{F}\cdot d\vec{r} = \int_0^{2\pi}\vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta)\,d\theta = \int_0^{2\pi}\langle 12\sin\theta, 4\cdot(-7), -24\cos\theta\rangle\cdot\langle -4\sin\theta, 4\cos\theta, 0\rangle\,d\theta =$$

$$= \int_0^{2\pi}(-48\sin^2\theta - 28\sin\theta)\,d\theta = -48\int_0^{2\pi}\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\,d\theta - 28\underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{0} =$$

$$-24\underbrace{\int_0^{2\pi}d\theta}_{2\pi} + 24\cdot\underbrace{\int_0^{2\pi}\cos 2\theta\,d\theta}_{0} = \boxed{-48\pi}$$

•Parametrize the surface of the paraboloid and compute the surface integral:

$$S = \{(x,y,z)\mid z = 9-x^2-y^2,\ z\ge -7\}$$

$\Updownarrow$

$\vec{r}(x,y) = \langle x, y, 9-x^2-y^2\rangle$ $\qquad 9-x^2-y^2\ge -7 \Leftrightarrow x^2+y^2\le 16$

$D = \{(x,y) : x^2+y^2\le 16\}$

$\vec{N}(x,y) = \pm\langle -z_x, -z_y, 1\rangle$

choose + as the normal is upward

$$\vec{N}(x,y) = \langle -z_x, -z_y, 1\rangle \underset{z=9-x^2-y^2}{=} \langle 2x, 2y, 1\rangle$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3y & 4z & -6x\end{vmatrix} = -4\hat{i} + 6\hat{j} - 3\hat{k} = \langle -4, 6, -3\rangle \Rightarrow$$

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \operatorname{curl}\vec{F}(\vec{r}(x,y))\cdot\vec{N}(x,y)\,dx\,dy =$$

$$= \iint_D\langle -4, 6, -3\rangle\cdot\langle 2x, 2y, 1\rangle\,dx\,dy = \iint_D(-8x+6y-3)\,dx\,dy \underset{\text{polar change}}{=}$$

$$= \int_0^{2\pi}\int_0^4(-8r\cos\theta + 6r\sin\theta - 3)\,r\,dr = \underbrace{\int_0^{2\pi}\cos\theta}_{0}\int_0^4(-8r^2)\,dr + \underbrace{\int_0^{2\pi}\sin\theta}_{0}\int_0^4 6r^2\,dr -$$

$$- 2\pi\int_0^4 3r\,dr = -6\pi\left.\frac{r^2}{2}\right|_0^4 = -3\pi\cdot 4^2 = \boxed{-48\pi}$$

THEOREM 3. *If* $\mathbf{F}$ *is a vector field defined on* $\mathbb{R}^3$ *whose component functions have continuous partial derivatives and* $\operatorname{curl}\mathbf{F} = \mathbf{0}$, *then* $\mathbf{F}$ *is a conservative vector field.*

SUMMARY: Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ *be a continuous vector field in* $\mathbb{R}^3$. (or in a simply connected domain E in $\mathbb{R}^3$

There exists f s.t. $\nabla f = \mathbf{F}$

$\int_{\breve{AB}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path

$\mathbf{F}$ is conservative in $\mathbb{R}^3$

$\operatorname{curl}\mathbf{F} = \mathbf{0}$

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C

not true in general if E is not simply connected