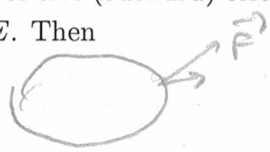
16.9: The Divergence Theorem

Let E be a simple solid region with the boundary surface S (which is a closed surface, i.e. without boundary) Let S be positively oriented (i.e.the orientation on S is outward that is, the unit normal vector $\hat{\mathbf{n}}$ is directed outward from E).

	Boundary
plain region D	∂D is a plane curve
surface S	∂S is a curve or union of two or more curves
solid E	∂E is a

The Divergence Theorem: Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\mathbf{F}$ be a continuous vector field on an open region that contains E. Then



$$\underbrace{\iint_S \mathbf{F}\cdot d\mathbf{S}}_{\text{flux}} = \iiint_E \underbrace{\mathrm{div}\mathbf{F}}_{\text{scalar field (a function)}}\, dV.$$

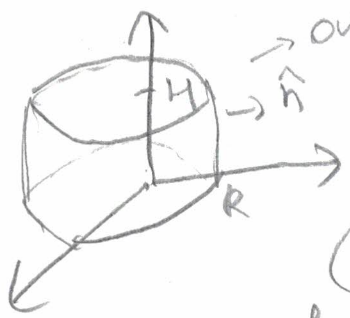
If the vector field is the field of the velocities of the fluid then divergence represents the rate of expansion fo the volume per unit volume ($\mathrm{div}\mathbf{F} > 0$ corresponds to expansion; $\mathrm{div}\mathbf{F} < 0$ corresponds to compression).

For sufficiently small ball E around a given point (x_0, y_0, z_0)

$$\mathrm{div}\vec{F}(x_0,y_0,z_0) \simeq \frac{\oiint \vec{F}\cdot d\vec{S}}{\iiint_E dV} = \frac{\text{Flux through } \partial E}{\text{Volume of } (E)}$$

The Divergence Theorem says that the divergence is the outgoing/ingoing flux per volume.

EXAMPLE 1. *Let $E = \{(x,y,z) : x^2+y^2 \le R^2, 0 \le z \le H\}$. Find the flux of the vector field $\mathbf{F} = \langle 1+x, 3+y, z-10\rangle$ over ∂E.*



$\to$ outward $\to \hat{n}$

$\iint_{\partial E} \vec{F}\cdot d\vec{S}$

$\downarrow$

direct calculation ☹ $\to$ we have to parametrize separately top, bottom, and the lateral part

∂E is closed

By the Divergence Thm ☺:

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E (1+1+1)\,dV = 3\iiint_E V = 3V(E) = 3V(E) = 3\pi R^2 H$$

REMARK 2. If $\mathbf{F} = \left\langle \frac{x}{3}, \frac{y}{3}, \frac{z}{3} \right\rangle$ then

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}\vec{F}}_{1}\,dV = Vol(E)$$

EXAMPLE 3. *Evaluate $I = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ if S is the boundary of*

(a) *ellipsoid* $E = \left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\right\}$ *and* $\mathbf{F} =$ something (actually it does not matter what)

$$I = \iint_{\partial E} \operatorname{curl} F\cdot d\vec{S} \xrightarrow[\text{Thm}]{\text{Divergence}} \iiint_E \underbrace{\operatorname{div}(\operatorname{curl}\vec{F})}_{\text{always } 0}\,dV = \boxed{0}$$

(b) *an arbitrary simple solid region E and F is an arbitrary continuous vector field.*

then $\iint_{\partial E} \operatorname{curl}\vec{F}\cdot d\vec{S} = 0$ (In item (a) we did not use that E is an ellipsoid

EXAMPLE 4. *Let E be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Evaluate* $I = \iint_S \langle x^3, 2xz^2, 3y^2 z\rangle \cdot d\mathbf{S}$ *if*

(a) *S is the boundary of the solid E.* $\Rightarrow$ we can apply the divergence theorem

$$I = \iint_{\partial E} \langle x^3, 2xz^2, 3y^2 z\rangle \cdot d\vec{S} =$$

$$= \iiint_E \underbrace{\operatorname{div}\langle x^3, 2xz^2, 3y^2 z\rangle}_{3x^2+0+3y^2}\, dx\,dy\,dz = 3\iiint_E (x^2+y^2)\,dx\,dy\,dz$$

$$= 3\iint_D \left(\int_0^{4-(x^2+y^2)} x^2+y^2\, dz\right) dx\,dy = 3\iint_D (x^2+y^2)(4-(x^2+y^2))\,dx\,dy$$

$D \to$ the projection of E on xy-plane, $D = \{(x,y) \mid x^2+y^2 \le 4\}$

$\underset{\text{change to polar}}{=}$ $$3\int_0^{2\pi}\int_0^2 r^2(4-r^2)\,r\,dr\,d\theta = 6\pi \int_0^2 (4r^3 - r^5)\,dr =$$

$$= 6\pi\left(\cancel{4}\frac{2^4}{\cancel{4}} - \frac{2^6}{6}\right) = \cancel{6}\pi\left(\frac{6\cdot 2^4 - 2^6}{\cancel{6}}\right) = (96-64)\pi = \boxed{32\pi}$$

(B) *S is the part of the paraboloid $z = 4 - x^2 - y^2$ between the planes $z = 0$ and $z = 4$.*

S is not a closed surface. If E is as in the previous item then $\partial E = S \cup$ the bottom disk. Denote this disk by S_0, $S_0 = \{(x,y,z) : z=0, x^2+y^2 \le 1\}$

Using part (A): $32\pi = \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_0} \vec{F}\cdot d\vec{S} \Rightarrow \iint_S \vec{F}\cdot d\vec{S} = 32\pi - \iint_{S_0}\vec{F}\cdot d\vec{S}$

$\iint_S$ → what we need

$\vec{n} = -\vec{k} = \langle 0,0,-1\rangle$

Parametrize S_0: $x=x, y=y, z=0$, $x^2+y^2 \le 4$ $\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = 32\pi -$

$$-\iint_{x^2+y^2\le 4} \underbrace{\langle x^3, 2x\cdot 0^2, 3y^2\cdot 0\rangle \cdot \langle 0,0,-1\rangle}_{0}\, dx\,dy = 32\pi - 0 = \boxed{32\pi}$$

EXAMPLE 5. *Verify the Divergence Theorem for the vector field* $\mathbf{F}(x,y,z)=\langle x^3,y^3,z^3\rangle$ *and* S, *which is the surface of the region enclosed by the cylinder* $x^2+y^2=1$ *and the planes* $z=1$ *and* $z=0$.

Divergence Thm: $\underbrace{\iint_S \vec F\cdot d\vec S}_{LHS} = \underbrace{\iiint_E \operatorname{div}\vec F\, dV}_{RHS}$

LHS $\oint\!\!\oint_S \vec F\cdot d\vec S = \iint_{S_1}\vec F\cdot d\vec S+\iint_{S_2}\vec F\cdot d\vec S+\iint_{S_3}\vec F\cdot d\vec S$

• Flux across S_1: parametrization of S_1: ~~x=~~ $x=x,\ y=y,\ z=1,$ $x^2+y^2\le 1$, $\vec N=\langle 0,0,1\rangle$ upward

$$\iint_{S_1}\vec F\cdot d\vec S=\iint_{x^2+y^2\le 1}\langle x^3,y^3,\underbrace{1^3}_{z=1}\rangle\cdot\langle 0,0,1\rangle\,dxdy=\iint_{x^2+y^2\le1}1\,dxdy = \text{Area of a unit disk}=\boxed{\pi}$$

• Flux across S_2: Parametrization of S_2: $x=x,\ y=y,\ z=0,$ $x^2+y^2\le 1$, $\vec N=\langle 0,0,-1\rangle$ downward

$$\iint_{S_2}\vec F\cdot d\vec S=\iint_{x^2+y^2\le 1}\underbrace{\langle x^3,y^3,\underbrace{0^3}_{z=0}\rangle\cdot\langle 0,0,-1\rangle}_{0}\,dxdy=\boxed{0}$$

• Flux across S_3: Parametrization of S_3 (use the cylindrical coordinates)

$x=\cos\theta,\ y=\sin\theta,\ z=z$ (here $r=1$),

$0\le\theta\le 2\pi,\ 0\le z\le 1 \Rightarrow \vec r(\theta,z)=\langle\cos\theta,\sin\theta,z\rangle$

$\vec r_\theta=\langle-\sin\theta,\cos\theta,0\rangle$, $\vec r_z=\langle 0,0,1\rangle$, $\vec N=\pm\vec r_\theta\times\vec r_z=\pm\begin{vmatrix}\hat i&\hat j&\hat k\\-\sin\theta&\cos\theta&0\\0&0&1\end{vmatrix}$ we choose the sign late[r]

$=\pm(\cos\theta\,\hat i+\sin\theta\,\hat j+0\,\hat k)=+\langle\cos\theta,\sin\theta,0\rangle$

$\vec N$ should be outword $\Rightarrow \vec N=\langle\cos\theta,\sin\theta,0\rangle\Rightarrow$

$$\iint_{S_3}\vec F\cdot d\vec S=\int_0^1\int_0^{2\pi}\langle\cos^3\theta,\sin^3\theta,z^3\rangle\cdot\langle\cos\theta,\sin\theta,0\rangle\,d\theta dz=$$

$$=\int_0^1\int_0^{2\pi}(\cos^4\theta+\sin^4\theta)\,d\theta dz=\int_0^{2\pi}\left(\left(\frac{1+\cos2\theta}{2}\right)^2+\left(\frac{1-\cos2\theta}{2}\right)^2\right)d\theta= \quad =1-\cos4\theta$$

$$=\frac14\int_0^{2\pi}\left(1+\cancel{2\cos2\theta}+\cos^22\theta+1-\cancel{2\cos2\theta}+\cos^22\theta\right)=\frac14\int_0^{2\pi}\left(2+2\cos^22\theta\right)d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}(3-\cos 4\theta)\,d\theta = \frac{3}{4}\cdot 2\pi - \underbrace{\int_0^{2\pi}\cos 4\theta\,d\theta}_{0} = \boxed{\frac{3}{2}\pi}$$

So $LHS = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = \pi + 0 + \frac{3}{2}\pi = \boxed{\frac{5\pi}{2}}$

RHS: $\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E \operatorname{div}\langle x^3, y^3, z^3\rangle\,dV =$

$$= \iiint_E 3(x^2+y^2+z^2)\,dx\,dy\,dz$$

Use cylindrical coordinates: $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$E = \{(x,y,z) \mid x^2+y^2 \le 1,\ 0 \le z \le 1\}$

$E^* = \{(r,\theta,z) \mid \underbrace{r^2 \le 1}_{0 \le r \le 1},\ 0 \le z \le 1,\ 0 \le \theta \le 2\pi\}$

$$\Rightarrow \iiint_E \operatorname{div}\vec{F}\,dV = 3\int_0^{2\pi}\int_0^1\int_0^1 \underbrace{(r^2+z^2)r}_{r^3+z^2r}\,dz\,dr\,d\theta =$$

$$= 6\pi\int_0^1\int_0^1 (r^3+rz^2)\,dz\,dr = 6\pi\int_0^1 \left(\left(r^3z + r\frac{z^3}{3}\right)\Big|_{z=0}^{1}\right)dr =$$

$$= 6\pi\int_0^1\left(r^3+\frac{r}{3}\right)dr = 6\pi\left(\frac{r^4}{4}+\frac{r^2}{6}\right)\Big|_{r=0}^{1} = 6\pi\left(\underbrace{\frac{1}{4}+\frac{1}{6}}_{\frac{3}{12}+\frac{2}{12}}\right) =$$

$$= 6\pi\cdot\frac{5}{12} = \boxed{\frac{5\pi}{2}} \Rightarrow LHS = RHS$$